Seminar Handout (Week 1)

Introduction to Nonlinear Wave Equations

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0. Notations.

$$\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$$

$$\nabla^2 = (\partial_1^2, \partial_1\partial_2, \cdots, \partial_{n-1}\partial_n, \partial_n^2),\ \nabla^3 =, \cdots, \cdots$$

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

C_0^j : C^j *functions with compact supports.*

1. Problem.

The nonlinear wave equations which shall be considered here are

$$y_{tt} - \Delta y = f(y_t, \nabla y, \nabla y_t, \nabla^2 y) := f(u, \nabla u) \tag{1}$$

with initial data

$$y(t=0) = y_0, \quad y_t(t=0) = y_1. \tag{2}$$

For convenience, let

$$u := (y_t, y_{x_1}, \cdots, y_{x_n}), \quad \nabla u := \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right) \quad u_0 := (y_1, \nabla y_0).$$

For example,

$$n = 1, \quad (u, \nabla u) = \left(y_t, \frac{\partial y}{\partial x_1}, \frac{\partial y_t}{\partial x_1}, \frac{\partial^2 y}{\partial x_1^2}\right)$$

$$n = 2, \quad (u, \nabla u) = \left(y_t, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial y_t}{\partial x_1}, \frac{\partial^2 y}{\partial x_1^2}, \frac{\partial^2 y}{\partial x_1 \partial x_2}, \frac{\partial y_t}{\partial x_2}, \frac{\partial^2 y}{\partial x_2 \partial x_1}, \frac{\partial^2 y}{\partial x_2^2}\right).$$

In the equation (1), we assume that f satisfies

$$f \in C^\infty(\mathbb{R}^{(n+1)^2}, \mathbb{R}), \tag{3}$$

$$\exists \alpha \in \mathbb{N} : \ f(u, \nabla u) = \mathcal{O}((|u| + |\nabla u|)^{\alpha+1} \ \ as \ \ |u| + |\nabla u| \to 0. \tag{4}$$

2. Theorems.

Theorem 1.1 *We assume (3) and (4) with* $\frac{1}{\alpha}(1+\frac{1}{\alpha}) < \frac{n-1}{2}$. *Then there exist an integer* $s_0 > \frac{n}{2}+1$ *and a* $\delta > 0$ *such that the following holds:*
If $u_0 = (y_1, \nabla y_0)$ *belongs to* $W^{s,2} \cap W^{s,p}$ *with* $s \geqslant s_0$ *and* $p = \frac{2\alpha+2}{2\alpha+1}$ *and* $||u_0||_{s,2} + ||u_0||_{s,p} < \delta$, *then there is a unique solution* $y \in C^2([0,\infty) \times \mathbb{R}^n)$ *of the initial value problem to the nonlinear wave equation (1) and (2).*

Example.

$$y_{tt} - \nabla \cdot \frac{\nabla y}{\sqrt{1+|\nabla y|^2}} = 0.$$

Question. Do quadratic nonlinearities ($\alpha = 1$) require n to be at least 6?
$\Rightarrow$ No. In fact,

$$\frac{1}{\alpha} < \frac{n-1}{2}.$$

Note. It is known that quadratic nonlinearities in $\mathbb{R}^3$ in general tend to develop singularities in finite time. Here, we introduce John's Blow up Theorem. Consider the nonlinear equation as

$$y_{tt} - \Delta y = |y|^p$$

where $y : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and $1 < p < \infty$. Then

(i) If $p < 1+\sqrt{2}$, then there exist solutions which are arbitrarily small (both in size and in support) and smooth at time zero, but which blow up in finite time.
(ii) If $p > 1+\sqrt{2}$, then for every initial data which is sufficiently small in size and support, and sufficiently smooth, one has a global solution (which goes to zero uniformly as $t \to \infty$).

3. Some Examples.

Example 1. *The global solutions may exist but also singularities may develop depending on the initial conditions.*

Consider the following problem.

$$y_{tt} - \Delta y = |\nabla y|^2 - y_t^2,$$

$$y(t=0) = 0, \;\; y_t(t=0) = h \in C^2(\mathbb{R}^3)$$

where $y = y(t,x),\ t \in \mathbb{R},\ x \in \mathbb{R}^3$.

Claim: $y(t,x) = \log(1 + \frac{t}{4\pi}\int_{S^2} h(x+t\zeta)d\zeta)$ is a solution. Also,
(i) y develops a singularity at (t_0, x_0) if $\frac{t_0}{4\pi}\int_{S^2} h(x_0+t_0\zeta)d\zeta = -1$.
(ii) y can be defined globally if $h(z) = \mathcal{O}(|z|^{-1})$ as $|z| \to \infty,\ ||\nabla h||_1 < 4\pi,\ ||h||_\infty < 1$.

Setting h: For example, if we set $h(x_1,x_2,x_3) = x_1^2 - x_2^2 + x_3^2$, then

$$y(t,x) = \log\left(1 + \frac{t}{4\pi}\int_{S^2} h(x+t\zeta)d\zeta\right) = \log\left(1 + t\left(\frac{t^2}{3} + x_1^2 - x_2^2 + x_3^2\right)\right).$$

So if $x_1 = x_3 = 1,\ x_2 = \sqrt{\frac{t_0^3+6t_0+3}{3t_0}}$ always develop a singularity at time t_0.

Example 2. *The solution itself and its first derivatives are bounded but that second derivatives may develop singularities in finite time.*

Consider the following problem.

$$y_{tt} = (1+y_x)^2 y_{xx}$$

$$y(t=0) = H,\ y_t(t=0) = -\frac{1}{2}(H')^2 - H'$$

where $y = y(t,x),\ t \geqslant 0,\ x \in \mathbb{R},\ H \in C_0^\infty(\mathbb{R})$ and $h := \min_{x\in\mathbb{R}} H''(x) < 0$.

Claim: $y(t,x) := \frac{t}{2}\theta^2(t,x) + H(x-(1+\theta(t,x))t)$ is a solution for $(t,x) \in [0,-1/h) \times \mathbb{R})$ where $\theta(t,x) = H'(x-(1+\theta(t,x))t)$. Here we will observe that y, y_x, y_t stay bounded in $[0,-1/h) \times \mathbb{R}$, but there is a sequence $\{t_n\} \subset [0,-1/h)$ with $\lim t_n = -1/h$ and a $x_0 \in \mathbb{R}$ such that $\lim y_{xx}(t_n, x_0) = -\infty$.

Setting H(x): Let $H(x) = e^{-1/x(1-x)}$ on $(0,1)$, $H(x) = 0$ otherwise, so that $H \in C_0^\infty(\mathbb{R})$ and $h := \min_{x\in\mathbb{R}} H''(x) = H''(1/2) = -\frac{32}{e^4} < 0$. Then the solution for $(t,x) \in [0,-1/h) \times \mathbb{R})$ is given by

$$y(t,x) = \frac{t}{2}\theta^2 + \exp\left(-\frac{1}{(x-(1+\theta)t)(1-x+(1+\theta)t)}\right)$$

where $\theta = \theta(t,x)$ satisfies

$$\exp\left(\frac{1}{(x-(1+\theta)t)(x-(1+\theta)t-1)}\right) = \frac{(x-(1+\theta)t)^2(x-(1+\theta)t-1)^2\theta}{1-2(x-(1+\theta)t)}.$$

References

[1] R. Racke, *Lectures on Nonlinear Evolution Equations*
[2] T. Tao, *John's blowup theorem for the nonlinear wave equation.*