Differential Equations II (240022, N. Hayashi)

R E P O R T

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1. Introduction

In this report, we study the global existence of solutions to nonlinear dispersive wave equations

$$\partial_t^2 u + \frac{1}{\rho^2}|\partial_x|^{2\rho} u \;=\; \lambda|\partial_t u|^{p-1}\partial_t u$$

in one space dimension where $0 < \rho \leqslant 2$, $\rho \neq 1$, $p > 3$ and $\lambda \in \mathbb{C}$.

In the second part, we introduce some L^1-L^∞ decay estimates for some partial differential equations as well as Schrödinger equations in n space dimension. For the Schrödinger equations, we extended the L^1-L^∞ estimate for the Schrödinger equations to n space dimension by the similar method in Hayashi [1], [2] for one space dimension which uses the free evolution group. And for the wave equations and the Klein-Gordon equations, the proof is similar to Wahl [3] which is given by the classical methods.

In the third part, we will prove the global existence of solutions to the nonlinear dispersive wave equations above which is given by Hayashi [2] using some estimates for the free evolution group. We will give some more details of Hayashi [2] there.

We introduce some notations here. Denote the Lebesgue space by

$$L^p = \{f \in S' : \|f\|_{L^p} < \infty\},\ \|f\|_{L^p} = \left(\int |f(x)|^p\, dx\right)^{1/p}$$

for $1 \leqslant p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup}|f(x)|$ for $p = \infty$. Also denote the Sobolev space by

$$W^{s,p} = \{f \in L^p : \|f\|_{W^{s,p}} < \infty\},\ \|f\|_{W^{s,p}} = \left(\sum_{|\alpha|\leqslant s} \|\nabla^\alpha f\|_{L^p}^p\right)^{1/p}$$

for nonnegative integer s and multi index α where $\nabla = (\partial_1, \cdots, \partial_n)$. And the weighted Sobolev space is

$$H_m^{s,p} = \{f \in S' : \|f\|_{H_m^{s,p}} = \|\langle i\nabla\rangle^s \langle x\rangle^m f\|_{L^p} < \infty\}$$

where $\langle x\rangle = \sqrt{1+|x|^2},\ s, m \in \mathbb{R}$. And we use the Fourier transform of f which is given by

$$\mathcal{F}f = \hat{f} = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} f(x)\, dx$$

and the inverse Fourier transform

$$\mathcal{F}^{-1} f = (2\pi)^{-n/2} \int e^{ix\cdot\xi} f(\xi)\, d\xi$$

and let $\Delta = \partial_1^2 + \cdots + \partial_n^2,\ (-\Delta)^{\alpha/2} = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}$.

2. L^1-L^∞ DECAY ESTIMATES OF SOME PDES.

In this section, we will derive some L^1-L^∞ time decay estimates for the solution of some linear partial differential equations such as **(1) Classical Schrödinger Equations**, **(2) Schrödinger Equations**, **(3) Wave Equations**, and **(4) Klein-Gordon Equations** in the n space dimensions. In the section (1) and (2), we use the free Schrödinger evolution group for the proof which is similar to Hayashi [1], [2]. In the section (3) and (4), we use the classical method given by Wahl [3].

(1) Classical schrödinger equations

We consider the following Schrödinger equation:

$$\partial_t u + i\Delta u = 0, \quad u(0,x) = u_0(x) \tag{2.1}$$

where $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. Taking the Fourier transform both sides of (2.1),

$$\partial_t \hat{u} - i|\xi|^2 \hat{u} = 0 \qquad \text{so that} \qquad \hat{u} = e^{it|\xi|^2}\hat{u_0}.$$

And taking the inverse Fourier transform both sides, we get

$$u \;=\; \mathcal{F}^{-1} e^{it|\xi|^2} \mathcal{F} u_0 \;=:\; \mathcal{W}(t) u_0$$

where the Schrödinger evolution group $\mathcal{W}(t) := \mathcal{F}^{-1} e^{it|\xi|^2}\mathcal{F}$. Then the following estimate holds.

Lemma 2.1 *For a function ϕ which is defined on $\mathbb{R} \times \mathbb{R}^n$, the estimate*

$$\|\mathcal{W}(t)\phi\|_{L^\infty} \;\leqslant\; Ct^{-\frac{n-\alpha}{2}} \|(-\Delta)^{\alpha/2}\phi\|_{L^1} \tag{2.2}$$

holds for all $t \neq 0$, provided that the right-hand side is finite, where $n-1 \leqslant \alpha < n$. Note that if we set $\alpha = n-1$, then

$$\|\mathcal{W}(t)\phi\|_{L^\infty} \;\leqslant\; Ct^{-\frac{1}{2}} \|(-\Delta)^{\frac{n-1}{2}}\phi\|_{L^1}$$

holds.

Proof. Note that $\mathcal{W}(t)\phi$ can be written by

$$\begin{aligned}
\mathcal{W}(t)\phi &= \mathcal{F}^{-1} e^{it|\xi|^2}\mathcal{F}\phi \;=\; \mathcal{F}^{-1} e^{it|\xi|^2}|\xi|^{-\alpha}\mathcal{F}\mathcal{F}^{-1}|\xi|^{\alpha}\mathcal{F}\phi \;=\; \mathcal{F}^{-1} e^{it|\xi|^2}|\xi|^{-\alpha}\mathcal{F}(-\Delta)^{\alpha/2}\phi \\
&= (2\pi)^{-\frac{n}{2}} (\mathcal{F}^{-1} e^{it|\xi|^2}|\xi|^{-\alpha}) * (-\Delta)^{\alpha/2}\phi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi + it|\xi|^2} |\xi|^{-\alpha}\, d\xi \right) (-\Delta_y)^{\alpha/2}\phi(y)\, dy \\
&= \int_{\mathbb{R}^n} K_\alpha(t, x-y)(-\Delta_y)^{\alpha/2}\phi(y)\, dy
\end{aligned}$$

where

$$K_\alpha(t,x) \;=\; \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it|\xi|^2 + ix\cdot\xi} |\xi|^{-\alpha}\, d\xi.$$

By changing the variable of integration $t^{\frac{1}{2}}\xi = \xi'$ we have

$$K_\alpha(t,x) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i|\xi'|^2+\frac{i}{\sqrt{t}}x\cdot\xi'}|\xi'|^{-\alpha}t^{-\frac{n-\alpha}{2}}\,d\xi' \;=\; t^{-\frac{n-\alpha}{2}}G_\alpha\left(\frac{x}{\sqrt{t}}\right)$$

where

$$G_\alpha(\eta) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i|\xi|^2+i\eta\cdot\xi}|\xi|^{-\alpha}\,d\xi.$$

So if we prove that $|G_\alpha(\eta)| \leqslant C$ for all $\eta \in \mathbb{R}^n$ then

$$|\mathcal{W}(t)\phi(t,x)| \;\leqslant\; Ct^{-\frac{n-\alpha}{2}}\int_{\mathbb{R}^n}(-\Delta_y)^{\alpha/2}\phi(y)\,dy \;=\; Ct^{-\frac{n-\alpha}{2}}\|(-\Delta)^{\alpha/2}\phi\|_{L^1}$$

thus the lemma follows. In fact this work is one of particular case of the next lemma, **Lemma 2.2**. So the inequality $|G_\alpha(\eta)| \leqslant C$ will be shown in the proof of **Lemma 2.2**. □

(2) Schrödinger equations

In this section we study the large time asymptotic behavior for the free evolution group of Schrödinger type $\mathcal{W}(t) = \mathcal{F}^{-1}\exp(\frac{it}{\rho}|\xi|^\rho)\mathcal{F}$ where $\rho \in (0,1)\cup(1,\infty)$. Consider the following Schrödinger equation:

$$\partial_t u - \frac{i}{\rho}(-\Delta)^{\rho/2}u = 0, \quad u(0,x) = u_0(x) \tag{2.3}$$

where $(t,x) \in \mathbb{R}\times\mathbb{R}^n$. Taking the Fourier transform both sides of (2.3),

$$\partial_t\hat{u} - \frac{i}{\rho}|\xi|^\rho\hat{u} = 0 \qquad \text{so that} \qquad \hat{u} = e^{\frac{it}{\rho}|\xi|^\rho}\hat{u_0}.$$

And taking the inverse Fourier transform both sides, we get

$$u \;=\; \mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}\mathcal{F}u_0 \;=:\; \mathcal{W}(t)u_0$$

where the Schrödinger evolution group $\mathcal{W}(t) := \mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}\mathcal{F}$. Then the following estimate holds.

Lemma 2.2 *Let $\rho \in (0,1)\cup(1,\infty)$. Then for a function ϕ which is defined on $\mathbb{R}\times\mathbb{R}^n$, the estimate*

$$\|\mathcal{W}(t)\phi\|_{L^\infty} \;\leqslant\; Ct^{-\frac{n-\alpha}{\rho}}\|(-\Delta)^{\alpha/2}\phi\|_{L^1} \tag{2.4}$$

holds for all $t \neq 0$, provided that the right-hand side is finite, where $n - \frac{\rho}{2} \leqslant \alpha < n$. Note that if we set $\alpha = n - \frac{\rho}{2}$ then

$$\|\mathcal{W}(t)\phi\|_{L^\infty} \;\leqslant\; Ct^{-\frac{1}{2}}\|(-\Delta)^{\frac{1}{2}(n-\frac{\rho}{2})}\|_{L^1}$$

holds.

Proof. Note that we can write $\mathcal{W}(t)\phi$ as

$$\begin{aligned}
\mathcal{W}(t)\phi &= \mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}\mathcal{F}\phi \\
&= \mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}|\xi|^{-\alpha}\mathcal{F}\mathcal{F}^{-1}|\xi|^{\alpha}\mathcal{F}\phi \\
&= \mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}\mathcal{F}(-\Delta)^{\alpha/2}\phi \\
&= (2\pi)^{-n/2}\left(\mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}|\xi|^{-\alpha}\right) * ((-\Delta)^{\alpha/2}\phi) \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi+\frac{it}{\rho}|\xi|^\rho}|\xi|^{-\alpha}\,d\xi\right)(-\Delta_y)^{\alpha/2}\phi(y)\,dy \\
&= \int_{\mathbb{R}^n} K_\alpha(t,x-y)(-\Delta_y)^{\alpha/2}\phi(y)\,dy
\end{aligned}$$

where the kernel

$$K_\alpha(t,x) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\frac{it}{\rho}|\xi|^\rho+ix\cdot\xi}|\xi|^{-\alpha}\,d\xi.$$

By changing the variable of integration $t^{\frac{1}{\rho}}\xi = \xi'$ we have

$$\begin{aligned}
K_\alpha(t,x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\rho}|\xi'|^\rho+ix\cdot t^{-\frac{1}{\rho}}\xi'}|\xi'|^{-\alpha}t^{-\frac{n-\alpha}{\rho}}\,d\xi' \\
&= t^{-\frac{n-\alpha}{\rho}}G_\alpha(t^{-\frac{1}{\rho}}x)
\end{aligned}$$

where

$$G_\alpha(\eta) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\rho}|\xi|^\rho+i\eta\cdot\xi}|\xi|^{-\alpha}\,d\xi.$$

Now Let us estimate G_α in the L^∞ norm. Note that for $\eta = (\eta_1,\cdots,\eta_n)$ and $\xi = (\xi_1,\cdots,\xi_n)$,

$$\begin{aligned}
G_\alpha(\pm\eta_1,\cdots,\pm\eta_n) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\rho}|\xi|^\rho+i(\pm\eta_1\xi_1\cdots\pm\eta_n\xi_n)}|\xi|^{-\alpha}\,d\xi \\
&= \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{\frac{i}{\rho}|\xi|^\rho+i(\pm\eta_1\xi_1\cdots\pm\eta_n\xi_n)}|\xi|^{-\alpha}\,d\xi_1\cdots d\xi_n \\
&= \frac{1}{(2\pi)^n}\int_{\mp\infty}^{\pm\infty}\cdots\int_{\mp\infty}^{\pm\infty} e^{\frac{i}{\rho}|\xi|^\rho+i(\eta_1\xi_1+\cdots+\eta_n\xi_n)}|\xi|^{-\alpha}\,(\pm d\xi_1)\cdots(\pm d\xi_n) \\
&= G_\alpha(\eta_1,\cdots,\eta_n).
\end{aligned}$$

So we only have to consider the case of $\eta_1,\cdots,\eta_n \geqslant 0$. For the simplicity we use the notation

$$\int_{\mathbb{R}^n_+} \;:=\; \int_0^\infty\cdots\int_0^\infty.$$

Then we can divide $G_\alpha(\eta)$ into

$$G_\alpha(\eta) \;=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+} e^{\frac{i}{\rho}|\xi|^\rho+i\xi\cdot\eta}|\xi|^{-\alpha}\,d\xi + \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+} e^{\frac{i}{\rho}|\xi|^\rho-i\xi\cdot\eta}|\xi|^{-\alpha}\,d\xi \;=:\; \text{I.1} + \text{I.2}.$$

Also note that for any vector $g, h \in \mathbb{R}^n$,

$$g\cdot h \;=\; |g||h|\epsilon$$

where $-1 \leqslant \epsilon = \epsilon_{gh} \leqslant 1$. Without loss of generality, let $\epsilon \neq 0$ because if $\epsilon = 0$ then we are in better situation since we have no inner product term such as $e^{\cdots\pm i\xi\cdot\eta}$ in I.1 and I.2. The first

integral I.1 can be easily estimated by rotating the contour of integration in the complex plane $\xi = \zeta e^{i\phi}$ such that $0 < \phi \leqslant \min\{\frac{\pi}{2\rho}, \frac{\pi}{2}\}$.

$$\begin{aligned}
|\text{I.1}| &= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_+} e^{\frac{i}{\rho}(\zeta_1^2+\cdots+\zeta_n^2)^{\rho/2} e^{i\rho\phi} + i\zeta\cdot\eta e^{i\phi}} |\zeta|^{-\alpha} e^{i\phi}\, d\zeta \right| \\
&\leqslant \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_+} e^{-\frac{1}{\rho}|\zeta|^\rho \sin\rho\phi - \zeta\cdot\eta\sin\phi} |\zeta|^{-\alpha}\, d\zeta \\
&\leqslant \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_+} e^{-\frac{2}{\pi}\phi(|\zeta|^\rho + \zeta\cdot\eta)} |\zeta|^{-\alpha}\, d\zeta \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_+} e^{-\frac{2}{\pi}\phi(|\zeta|^\rho + |\zeta||\eta|\epsilon)} |\zeta|^{-\alpha}\, d\zeta \\
&= \frac{1}{(2\pi)^n} \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-\frac{2}{\pi}\phi(r^\rho + r|\eta|\epsilon)} r^{-\alpha} r^{n-1} \sin^{n-2}(\theta_1)\cdots\sin(\theta_{n-2})\, dr d\theta_1 \cdots d\theta_{n-1} \\
&= \frac{1}{(2\pi)^n} \int_0^\infty e^{-\frac{2}{\pi}\phi(r^\rho + r|\eta|\epsilon)} r^{-\alpha+n-1}\, dr \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2}\, d\theta_1 \cdots d\theta_{n-1} \\
&\leqslant C \int_0^\infty e^{-\frac{2}{\pi}\phi(r^\rho + r|\eta|\epsilon)} r^{-\alpha+n-1}\, dr
\end{aligned}$$

Without loss of generality, assume that $1 \geqslant \epsilon > 0$ (if $-1 \leqslant \epsilon < 0$ then I.1 can be incorporated into the term I.2 below). For the 5th equality, we applied n-dimensional spherical coordinate system with $|\zeta| = r$, see **Appendix**. In fact this spherical coordinate system is available for $n \geqslant 2$, so without loss of generality, assume that $n \geqslant 2$(If $n = 1$ then we have easier calculations by the similar way). Therefore we finally estimate I.1 as

$$|\text{I.1}| \leqslant C$$

provided for all $\eta \in \mathbb{R}^n_+,\ \rho > 0,\ \alpha < n$.

The second integral I.2 for the case of $\rho > 1$ can be estimated by the similar way as I.1 if $0 \leqslant |\eta| \leqslant \epsilon^{-1}$. Again rotating the contour of integration in the complex plane $\xi = \zeta e^{i\phi}$ we get

$$|\text{I.2}| \leqslant C \int_0^\infty e^{-\frac{2}{\pi}\phi r^\rho + r|\eta|\epsilon\sin\phi} r^{-\alpha+n-1}\, dr \leqslant C \tag{2.5}$$

provided for $|\eta| \leqslant \epsilon^{-1},\ \rho > 1,\ \alpha < n$.

Now consider I.2 $= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_+} \exp(\frac{i}{\rho}|\xi|^\rho - i|\xi||\eta|\epsilon) |\xi|^{-\alpha}\, d\xi$ for the case that $\epsilon|\eta| \geqslant 1$ or $|\eta| \geqslant \epsilon^{-1}$. Without loss of generality, assume that $1 \geqslant \epsilon > 0$ (if $-1 \leqslant \epsilon < 0$ then I.2 can be incorporated into the term I.1). For the simplicity define H as

$$H(|\xi|) := -|\xi||\eta|\epsilon + \frac{1}{\rho}|\xi|^\rho = -|\xi|\mu^{\rho-1} + \frac{1}{\rho}|\xi|^\rho$$

where $\mu := |\eta|^{\frac{1}{\rho-1}} \epsilon^{\frac{1}{\rho-1}}$ so that

$$\partial_{|\xi|} H(|\xi|) = -\mu^{\rho-1} + |\xi|^{\rho-1} = 0$$

if $\mu = |\xi|$. Also note that

$$e^{iH(|\xi|)} = \frac{1}{i\partial_{|\xi|}H(|\xi|)} \partial_{|\xi|} e^{iH(|\xi|)} = \frac{i}{\mu^{\rho-1} - |\xi|^{\rho-1}} \partial_{|\xi|} e^{iH(|\xi|)}.$$

So we will split the integral I.2 into the two parts:

$$\begin{aligned}\text{I.2} &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+} e^{iH(|\xi|)}|\xi|^{-\alpha}\,d\xi \\ &= \frac{1}{(2\pi)^n}\int_{|\xi|\leqslant 2\mu} |\xi|^{-\alpha}e^{iH(|\xi|)}\,d\xi + \frac{1}{(2\pi)^n}\int_{|\xi|\geqslant 2\mu} |\xi|^{-\alpha}e^{iH(|\xi|)}\,d\xi.\end{aligned}$$

Applying n-dimensional spherical coordinate system with $|\xi| = r$ again,

$$\text{I.2} = C\int_0^{2\mu} r^{-\alpha}e^{iH(r)}r^{n-1}\,dr + C\int_{2\mu}^{\infty} r^{-\alpha}e^{iH(r)}r^{n-1}\,dr =: \text{I.2.a} + \text{I.2.b}.$$

Now consider I.2.b. Then the integration by parts yields

$$\begin{aligned}\text{I.2.b} &= C\int_{2\mu}^{\infty} r^{-\alpha+n-1}e^{iH(r)}\,dr \\ &= C\int_{2\mu}^{\infty} \frac{r^{-\alpha+n-1}}{i\partial_r H(r)}\partial_r e^{iH(r)}\,dr \\ &= C\int_{2\mu}^{\infty} \partial_r\left(\frac{r^{-\alpha+n-1}}{i\partial_r H(r)}e^{iH(r)}\right)\,dr - C\int_{2\mu}^{\infty}\partial_r\left(\frac{r^{-\alpha+n-1}}{i\partial_r H(r)}\right)e^{iH(r)}\,dr \\ &= C\int_{2\mu}^{\infty} \partial_r\left(\frac{ir^{-\alpha+n-1}}{\mu^{\rho-1}-r^{\rho-1}}e^{i(\frac{1}{\rho}r^\rho - r\mu^{\rho-1})}\right)\,dr - C\int_{2\mu}^{\infty}\partial_r\left(\frac{ir^{-\alpha+n-1}}{\mu^{\rho-1}-r^{\rho-1}}\right)e^{i(\frac{1}{\rho}r^\rho - r\mu^{\rho-1})}\,dr \\ &= C\frac{(2\mu)^{-\alpha+n-1}}{(2\mu)^{\rho-1}-\mu^{\rho-1}}e^{i\cdots} - C\int_{2\mu}^{\infty}\left(\frac{i(-\alpha+n-1)r^{-\alpha+n-2}}{\mu^{\rho-1}-r^{\rho-1}} + \frac{i(\rho-1)r^{-\alpha+n-3+\rho}}{(\mu^{\rho-1}-r^{\rho-1})^2}\right)e^{i\cdots}\end{aligned}$$

so that

$$\begin{aligned}|\text{I.2.b}| &\leqslant C\left|\frac{(2\mu)^{-\alpha+n-1}}{(2\mu)^{\rho-1}-\mu^{\rho-1}}\right| + C\int_{2\mu}^{\infty}\left|\frac{(-\alpha+n-1)r^{-\alpha+n-2}}{\mu^{\rho-1}-r^{\rho-1}}\right| + \left|\frac{(\rho-1)r^{-\alpha+n-3+\rho}}{(\mu^{\rho-1}-r^{\rho-1})^2}\right|\,dr \\ &\leqslant C\mu^{-\alpha+n-\rho} + C\int_{2\mu}^{\infty}\left|\frac{2(-\alpha+n-1)r^{-\alpha+n-2}}{r^{\rho-1}}\right| + \left|\frac{4(\rho-1)r^{-\alpha+n-3+\rho}}{r^{2\rho-2}}\right|\,dr \\ &\leqslant C\mu^{-\alpha-\rho+n} + C\int_{2\mu}^{\infty} r^{-\alpha-\rho+n-1}\,dr \\ &\leqslant C\mu^{-\alpha-\rho+n} \\ &\leqslant C\end{aligned}$$

provided for $|\eta| \geqslant \epsilon^{-1}$, $\rho > 1,\ n > \alpha > n-\rho$ ($\because\ -\alpha-\rho+n-1 < -1,\ -\alpha-\rho+n < 0$).

Now consider I.2.a. Then the integration by parts yields

$$\begin{aligned}\text{I.2.a} &= C\int_0^{2\mu} r^{-\alpha+n-1}e^{iH(r)}\,dr \\ &= C\int_0^{2\mu}\frac{r^{-\alpha+n-1}\partial_r((r-\mu)e^{iH(r)})}{1+(r-\mu)i\partial_r H(r)}\,dr \\ &= C\int_0^{2\mu}\frac{r^{-\alpha+n-1}\partial_r((r-\mu)e^{iH(r)})}{1-i(r-\mu)(\mu^{\rho-1}-r^{\rho-1})}\,dr \\ &= C\int_0^{2\mu}\partial_r\left(\frac{r^{-\alpha+n-1}(r-\mu)e^{iH(r)}}{1-i(r-\mu)(\mu^{\rho-1}-r^{\rho-1})}\right)\,dr \\ &\qquad -C\int_0^{2\mu}\left(\partial_r\frac{r^{-\alpha+n-1}}{1-i(r-\mu)(\mu^{\rho-1}-r^{\rho-1})}\right)(r-\mu)e^{iH(r)}\,dr \\ &=: \text{I.2.a.i} + \text{I.2.a.ii}.\end{aligned}$$

For I.2.a.i, we have

$$|\text{I.2.a.i}| \;\leqslant\; C\left|\frac{2^{-\alpha+n-1}\mu^{-\alpha+n-1}}{1-i\mu^{\rho}(1-2^{\rho-1})}\right| \;\leqslant\; C\mu^{-\alpha-\rho+n}.$$

For I.2.a.ii, by the mean value theorem, we have

$$\begin{aligned}
|\text{I.2.a.ii}| &\leqslant C\int_0^{2\mu}\left|\frac{(-\alpha+n-1)r^{-\alpha+n-2}(r-\mu)}{1-i(r-\mu)(\mu^{\rho-1}-r^{\rho-1})}\right| dr+\cdots\\
&\leqslant C\mu^{-\alpha+n-1}\int_0^{2\mu}\frac{dr}{1+(r-\mu)^2\mu^{\rho-2}}+\cdots\\
&\leqslant C\mu^{-\alpha+n-1}\int_{-\mu^{\rho/2}}^{\mu^{\rho/2}}\frac{\mu^{\frac{-\rho+2}{2}}}{1+r'^2}\,dr' \text{ (substitution: } r'=(r-\mu)\mu^{\frac{\rho-2}{2}})\\
&\leqslant C\mu^{-\alpha-\frac{\rho}{2}+n}
\end{aligned}$$

Therefore we have

$$|\text{I.2.a}| \;\leqslant\; C\mu^{-\alpha-\rho+n}+C\mu^{-\alpha-\frac{\rho}{2}+n} \;\leqslant\; C$$

provided for $|\eta|\geqslant\epsilon^{-1},\ \rho>1,\ n>\alpha\geqslant n-\frac{\rho}{2}$.

Combining these results we have

$$|\text{I.1}| \;\leqslant\; C \quad (\text{if } \eta\in\mathbb{R}^n_+,\ \rho>0,\ \alpha<n),$$

$$|\text{I.2}| \;\leqslant\; C \quad (\text{if } \eta\in\mathbb{R}^n_+,\ \rho>1,\ n-\frac{\rho}{2}\leqslant\alpha<n)$$

so we only have to estimate I.2 when $0<\rho<1$. But by the change of variables $|\eta|=m,\ r=\frac{1}{m}\rho^{\frac{1}{\rho}}\ell^{\frac{1}{\rho}},\ \nu=m^{-\rho}$, we have

$$\begin{aligned}
\text{I.2} &= C\int_0^{\infty} e^{\frac{i}{\rho}r^{\rho}-ir|\eta|\epsilon}r^{-\alpha+n-1}\,dr\\
&= C\int_0^{\infty} e^{i\nu\ell-i\rho^{\frac{1}{\rho}}\ell^{\frac{1}{\rho}}\epsilon}m^{\alpha-n+1}\rho^{\frac{-\alpha+n-1}{\rho}}\ell^{\frac{-\alpha+n-1}{\rho}}m^{-1}\rho^{\frac{1}{\rho}-1}\ell^{\frac{1}{\rho}-1}\,d\ell\\
&= C\rho^{\frac{1}{\rho}-1}\nu^{1-\alpha'}\int_0^{\infty} e^{-i\rho^{\frac{1}{\rho}}\ell^{\rho'}\epsilon+i\nu\ell}\ell^{-\alpha'}\,d\ell
\end{aligned}$$

with $\rho'=\frac{1}{\rho}>1,\ \alpha'=1-\frac{n-\alpha}{\rho}<1$. So by the similar way as above, we have

$$|\text{I.2}| \;\leqslant\; C$$

provided for $\eta\in\mathbb{R}^n_+,\ 0<\rho<1,\ n-\frac{\rho}{2}\leqslant\alpha<n$.

Combining all these results, we finally get

$$\begin{aligned}
|\mathcal{W}(t)\phi(t,x)| &\leqslant \int_{\mathbb{R}^n}|K_\alpha(t,x-y)(-\Delta_y)^{\alpha/2}\phi(y)|\,dy\\
&\leqslant Ct^{-\frac{n-\alpha}{\rho}}\int_{\mathbb{R}^n}\left|(-\Delta_y)^{\alpha/2}\phi(y)\right|\,dy \;=\; Ct^{-\frac{n-\alpha}{\rho}}\|(-\Delta_y)^{\alpha/2}\phi\|_{L^1}
\end{aligned}$$

for all $t\neq 0$ where $\rho\in(0,1)\cup(1,\infty)$ and $n-\frac{\rho}{2}\leqslant\alpha<n$ which proves **Lemma 2.2**. □

(3) Wave equations

Consider the wave equation

$$v_{tt} - \Delta v = 0 \text{ in } [0,\infty) \times \mathbb{R}^n \quad \text{with} \quad v(t=0) = \phi,\ \partial_t v(t=0) = \psi \tag{2.6}$$

where $\phi \in C_0^n(\mathbb{R}^n),\ \psi \in C_0^{n-1}(\mathbb{R}^n)$. And let v be a solution of (2.6). Then the following estimate holds.

Lemma 2.3 *There exists C such that for $t \geqslant 0$,*

$$\|v(t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{n-1}{2}}(\|\phi\|_{W^{n,1}} + \|\psi\|_{W^{n-1,1}}) \tag{2.7}$$

holds for all $n \in \mathbb{N}$.

Proof. For the simplicity denote $\|\cdot\|_p = \|\cdot\|_{L^p}$ and $\|\cdot\|_{s,p} = \|\cdot\|_{W^{s,p}}$. We will prove this for odd n, the similar proof can be applied for even n. If $n = 1$, then (2.7) is straight forward by d'Alembert's formula. Consider $n \geqslant 3$. Then for the equation (2.6), v is given by

$$v(t,x) = \sum_{j=0}^{\frac{n-3}{2}} (j+1) a_j t^j (\partial_t^j Q_1)(t,x) + t \sum_{j=0}^{\frac{n-3}{2}} a_j t^j \left((\partial_t^{j+1} Q_1)(t,x) + (\partial_t^j Q_2)(t,x) \right) \tag{2.8}$$

where $a_j = a_j(n)$ and

$$Q_1(t,x) := \frac{1}{w_n} \int_{S^{n-1}} \phi(x + t\xi) d\xi,$$
$$Q_2(t,x) := \frac{1}{w_n} \int_{S^{n-1}} \psi(x + t\xi) d\xi.$$

Here S^{n-1} denotes the unit-sphere in $\mathbb{R}^n$, w_n is its surface, $d\xi$ is its surface-element. Also we often use the following formula

$$\int_{S^{n-1}} \xi_j f(t\xi) d\xi = t^{-(n-1)} \int_{K_t(0)} (\partial_j f)(y) dy$$

for $f \in C^1(K_t(0)),\ t > 0,\ K_t(0) := \{y \in \mathbb{R}^n \mid |y| \leqslant t\}$.

(i) $t \geqslant 1$
For $j \geqslant 1$, we have

$$\begin{aligned}
|\partial_t^j Q_1| &= \frac{1}{w_n} \left| \int_{S^{n-1}} \partial_t^j \phi(x + t\xi) d\xi \right| \\
&= \frac{1}{w_n} \left| \sum_{|\alpha|=j} \int_{S^{n-1}} (\nabla^\alpha \phi)(x + t\xi) \xi^\alpha d\xi \right| \\
&\leqslant \frac{1}{w_n} \sum_{|\alpha|=j} \sum_{|\beta|=1, \alpha \geqslant \beta} t^{-|\alpha-\beta|} \left| \int_{S^{n-1}} (\nabla^\alpha \phi)(x + t\xi)(t\xi)^{\alpha-\beta} \xi^\beta d\xi \right| \\
&= \frac{1}{w_n} \sum_{|\alpha|=j, |\beta|=1, \alpha \geqslant \beta} t^{-|\alpha-\beta|-(n-1)} \left| \int_{K_t(0)} (\nabla^\alpha \nabla^\beta \phi)(x+y) y^{\alpha-\beta} dy \right| \\
&\leqslant \frac{1}{w_n} \sum_{|\alpha|=j+1, |\gamma|=|\alpha|-2} t^{-(n-1)} t^{-|\gamma|} \left| \int_{K_t(0)} (\nabla^\alpha \phi)(x+y) y^\gamma dy \right|
\end{aligned}$$

and similarly,

$$|\partial_t^j Q_2| \leqslant \frac{1}{w_n} \sum_{|\alpha|=j+1,|\gamma|=|\alpha|-2} t^{-(n-1)} t^{-|\gamma|} \left| \int_{K_t(0)} (\nabla^\alpha \psi)(x+y) y^\gamma dy \right|.$$

Now consider the integral $t^{-|\gamma|} \int_{K_t(0)} (\nabla^\alpha \phi)(x+y) y^\gamma dy$. Let $\lambda, \mu \geqslant 0$ such that $\mu < 1$ and $\lambda + \mu \leqslant 1$. Since $(1-\mu) + (1-\lambda) = 1 + (1-\lambda-\mu)$, by the convolution inequality,

$$\begin{aligned}
\left\| t^{-|\gamma|} \int_{K_t(0)} (\nabla^\alpha \phi)(\cdot + y) y^\gamma dy \right\|_{\frac{1}{1-\lambda-\mu}} &\leqslant t^{-|\gamma|} \|\nabla^\alpha \phi\|_{\frac{1}{1-\mu}} \left(\int_{K_t(0)} |y^\gamma|^{\frac{1}{1-\lambda}} dy \right)^{1-\lambda} \\
&\leqslant t^{-|\gamma|} \|\nabla^\alpha \phi\|_{\frac{1}{1-\mu}} \left(\int_{K_t(0)} t^{\frac{|\gamma|}{1-\lambda}} dy \right)^{1-\lambda} \\
&\leqslant t^{-|\gamma|} \|\nabla^\alpha \phi\|_{\frac{1}{1-\mu}} (C t^{n+\frac{|\gamma|}{1-\lambda}})^{1-\lambda} \\
&= C t^{n(1-\lambda)} \|\nabla^\alpha \phi\|_{\frac{1}{1-\mu}}.
\end{aligned}$$

Thus for $j \geqslant 1$,

$$\begin{aligned}
\|\partial_t^j Q_1(t)\|_{\frac{1}{1-\lambda-\mu}} &\leqslant \frac{1}{w_n} \sum_{|\alpha|=j+1,|\gamma|=|\alpha|-2} t^{-(n-1)} t^{-|\gamma|} \left\| \int_{K_t(0)} (\nabla^\alpha \phi)(\cdot + y) y^\gamma dy \right\|_{\frac{1}{1-\lambda-\mu}} \\
&\leqslant C \sum_{|\alpha|=j+1,|\gamma|=|\alpha|-2} t^{-(n-1)} t^{n(1-\lambda)} \|\nabla^\alpha \phi\|_{\frac{1}{1-\mu}} \\
&\leqslant C \sum_{|\alpha|=j+1} t^{1-n\lambda} \|\nabla^\alpha \phi\|_{\frac{1}{1-\mu}}
\end{aligned}$$

and

$$\|\partial_t^j Q_2(t)\|_{\frac{1}{1-\lambda-\mu}} \leqslant C \sum_{|\alpha|=j+1} t^{1-n\lambda} \|\nabla^\alpha \psi\|_{\frac{1}{1-\mu}}.$$

Also because

$$|Q_1| \leqslant \frac{1}{t w_n} \left| \sum_{j=1}^n \int_{S^{n-1}} \phi(x+t\xi) t \xi_j \xi_j d\xi \right| \leqslant C t^{-n} \sum_{|\alpha|,|\gamma|\leqslant 1} \left| \int_{K_t(0)} (\nabla^\alpha \phi)(x+y) y^\gamma dy \right|,$$

by the convolution inequality above, we have

$$\begin{aligned}
\|Q_1(t)\|_{\frac{1}{1-\lambda-\mu}} &\leqslant C t^{-n} \sum_{|\alpha|,|\gamma|\leqslant 1} \left\| \int_{K_t(0)} (\nabla^\alpha \phi)(\cdot + y) y^\gamma dy \right\|_{\frac{1}{1-\lambda-\mu}} \\
&\leqslant C t^{-n} \sum_{|\alpha|,|\gamma|\leqslant 1} t^{|\gamma|} t^{n(1-\lambda)} \|\nabla^\alpha \phi\|_{\frac{1}{1-\mu}} \\
&= C t^{-n\lambda} \sum_{|\alpha|,|\gamma|\leqslant 1} t^{|\gamma|} \|\nabla^\alpha \phi\|_{\frac{1}{1-\mu}} \\
&\leqslant C t^{1-n\lambda} \left(\|\phi\|_{\frac{1}{1-\mu}} + \sum_{|\alpha|=1} \|\nabla^\alpha \phi\|_{\frac{1}{1-\mu}} \right)
\end{aligned}$$

and

$$\|Q_2(t)\|_{\frac{1}{1-\lambda-\mu}} \leqslant C t^{1-n\lambda} \left(\|\psi\|_{\frac{1}{1-\mu}} + \sum_{|\alpha|=1} \|\nabla^\alpha \psi\|_{\frac{1}{1-\mu}} \right).$$

For the simplicity, let $\frac{1}{1-\lambda-\mu} =: z$. Combining all these results,

$$\begin{aligned}
\|v(t)\|_z &\leqslant C\left[\|Q_1(t)\|_z + t\|Q_2(t)\|_z + \sum_{j=1}^{\frac{n-3}{2}} t^j \|\partial_t^j Q_1(t)\|_z + t\sum_{j=1}^{\frac{n-3}{2}} t^j \left(\|\partial_t^{j+1} Q_1(t)\|_z + \|\partial_t^j Q_2(t)\|_z\right)\right] \\
&\leqslant C\Bigg[t^{1-n\lambda}\left(\|\phi\|_{\frac{1}{1-\mu}} + \sum_{|\alpha|=1}\|\nabla^\alpha\phi\|_{\frac{1}{1-\mu}}\right) + t^{2-n\lambda}\left(\|\psi\|_{\frac{1}{1-\mu}} + \sum_{|\alpha|=1}\|\nabla^\alpha\psi\|_{\frac{1}{1-\mu}}\right) + \\
&\quad \sum_{j=1}^{\frac{n-3}{2}} t^j \sum_{|\alpha|=j+1} t^{1-n\lambda}\|\nabla^\alpha\phi\|_{\frac{1}{1-\mu}} + t\sum_{j=1}^{\frac{n-3}{2}} t^j\left(\sum_{|\alpha|=j+2} t^{1-n\lambda}\|\nabla^\alpha\phi\|_{\frac{1}{1-\mu}} + \sum_{|\alpha|=j+1} t^{1-n\lambda}\|\nabla^\alpha\psi\|_{\frac{1}{1-\mu}}\right)\Bigg] \\
&\leqslant C\left[t^{1-n\lambda}\|\phi\|_{\frac{1}{1-\mu}} + t^{2-n\lambda}\|\psi\|_{\frac{1}{1-\mu}} + t^{\frac{1}{2}+\frac{n}{2}-n\lambda}\left(\sum_{1\leqslant|\alpha|\leqslant\frac{n-3}{2}+2}\|\nabla^\alpha\phi\|_{\frac{1}{1-\mu}} + \sum_{1\leqslant|\alpha|\leqslant\frac{n-3}{2}+1}\|\nabla^\alpha\psi\|_{\frac{1}{1-\mu}}\right)\right] \\
&\leqslant Ct^{\frac{n}{2}+\frac{1}{2}-n\lambda}\left(t^{-\frac{n-1}{2}}\|\phi\|_{\frac{1}{1-\mu}} + t^{-\frac{n-3}{2}}\|\psi\|_{\frac{1}{1-\mu}} + \sum_{1\leqslant|\alpha|\leqslant\frac{n+1}{2}}\|\nabla^\alpha\phi\|_{\frac{1}{1-\mu}} + \sum_{1\leqslant|\alpha|\leqslant\frac{n-1}{2}}\|\nabla^\alpha\psi\|_{\frac{1}{1-\mu}}\right).
\end{aligned}$$

Thus if we set $\mu = 0,\ \lambda = 1$ then

$$\|v(t)\|_\infty \leqslant Ct^{-\frac{n-1}{2}}\left(\|\phi\|_{\frac{n+1}{2},1} + \|\psi\|_{\frac{n-1}{2},1}\right)$$

for $t \geqslant 1$.

(ii) $0 \leqslant t \leqslant 1$
Consider the term $M = M(t,k,\alpha,h,x) := t^k \int_{S^{n-1}} (\nabla^\alpha h)(x+t\xi)\xi^\alpha d\xi$ for $h = \phi, \psi$ and $k \geqslant 0$.
For $m \in \mathbb{N}_0$,

$$\begin{aligned}
|M| &= t^k\left|\int_{S^{n-1}}\int_t^\infty \frac{(s-t)^m}{m!}\frac{d^{m+1}}{ds^{m+1}}\nabla^\alpha h(x+s\xi)\xi^\alpha ds d\xi\right| \\
&= \frac{t^k}{m!}\left|\int_{S^{n-1}}\int_t^\infty \frac{(s-t)^m(s\xi)^\alpha}{s^{|\alpha|+m+1}}\underbrace{\nabla(\nabla(\nabla(\cdots\nabla}_{m+1\text{ times}}(\nabla^\alpha h(x+s\xi))\underbrace{\cdot s\xi\cdots)\cdot s\xi)\cdot s\xi}_{m+1\text{ times}} ds d\xi\right| \\
&= \frac{t^k}{m!}\left|\int_{|y|=s}\int_t^\infty \frac{(s-t)^m y^\alpha}{s^{|\alpha|+m+1}}\nabla(\nabla(\nabla(\cdots\nabla(\nabla^\alpha h(x+y))\cdot y\cdots)\cdot y)\cdot y)\cdot y ds\frac{dy}{s^{n-1}}\right| \\
&= \frac{t^k}{m!}\left|\int_{|y|>t}\frac{(|y|-t)^m y^\alpha}{|y|^{|\alpha|+n+m}}\sum_{i_1,\cdots,i_{m+1}=1}^n y_{i_1}\cdots y_{i_{m+1}}\left(\partial_{i_1}\cdots\partial_{i_{m+1}}\nabla^\alpha h\right)(x+y)dy\right| \\
&\leqslant \frac{t^k}{m!}\int_{|y|>t}\frac{(|y|-t)^m y^\alpha}{|y|^{|\alpha|+n-1}}\sum_{i_1,\cdots,i_{m+1}=1}^n\left|\left(\partial_{i_1}\cdots\partial_{i_{m+1}}\nabla^\alpha h\right)(x+y)\right|dy \\
&\leqslant C\int_{|y|>t}\frac{|y|^{m+k}}{|y|^{n-1}}\sum_{i_1,\cdots,i_{m+1}=1}^n\left|\left(\partial_{i_1}\cdots\partial_{i_{m+1}}\nabla^\alpha h\right)(x+y)\right|dy \\
&= C\sum_{i_1,\cdots,i_{m+1}=1}^n\int_{|y|>t}\left|\left(\partial_{i_1}\cdots\partial_{i_{m+1}}\nabla^\alpha h\right)(x+y)\right|dy \quad (\text{if } m = n-1-k).
\end{aligned}$$

Now consider (2.8). Then we have

$$|v(t,x)| \leqslant A + B + C'$$

where

$$\begin{aligned} A &= C\sum_{j=0}^{\frac{n-3}{2}} \sum_{|\alpha|=j} t^j \left| \int_{S^{n-1}} (\nabla^\alpha \phi)(x+t\xi)\xi^\alpha d\xi \right| \\ &\leqslant C\sum_{j=0}^{\frac{n-3}{2}} \sum_{|\alpha|=j} \sum_{i_1,\cdots,i_{n-j}=1}^{n} \int_{|y|>t} \left| \left(\partial_{i_1}\cdots\partial_{i_{n-j}}\nabla^\alpha\phi\right)(x+y)\right| dy \ \leqslant\ C\|\phi\|_{n,1}, \end{aligned}$$

$$\begin{aligned} B &= C\sum_{j=0}^{\frac{n-3}{2}} \sum_{|\alpha|=j+1} t^{j+1} \left| \int_{S^{n-1}} (\nabla^\alpha \phi)(x+t\xi)\xi^\alpha d\xi \right| \\ &\leqslant C\sum_{j=0}^{\frac{n-3}{2}} \sum_{|\alpha|=j+1} \sum_{i_1,\cdots,i_{n-j-1}=1}^{n} \int_{|y|>t} \left| \left(\partial_{i_1}\cdots\partial_{i_{n-j-1}}\nabla^\alpha\phi\right)(x+y)\right| dy \ \leqslant\ C\|\phi\|_{n,1}, \end{aligned}$$

$$\begin{aligned} C' &= C\sum_{j=0}^{\frac{n-3}{2}} \sum_{|\alpha|=j} t^{j+1} \left| \int_{S^{n-1}} (\nabla^\alpha \psi)(x+t\xi)\xi^\alpha d\xi \right| \\ &\leqslant C\sum_{j=0}^{\frac{n-3}{2}} \sum_{|\alpha|=j} \sum_{i_1,\cdots,i_{n-j-1}=1}^{n} \int_{|y|>t} \left| \left(\partial_{i_1}\cdots\partial_{i_{n-j-1}}\nabla^\alpha\psi\right)(x+y)\right| dy \ \leqslant\ C\|\psi\|_{n-1,1}. \end{aligned}$$

Therefore we have

$$\|v(t)\|_\infty \leqslant C(\|\phi\|_{n,1} + \|\psi\|_{n-1,1})$$

for $0 \leqslant t \leqslant 1$. Summarizing the estimates (i) $t \geqslant 1$ and (ii) $0 \leqslant t \leqslant 1$, we have the desired estimates for $t \geqslant 0$,

$$\|v(t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{n-1}{2}}(\|\phi\|_{W^{n,1}} + \|\psi\|_{W^{n-1,1}})$$

for odd dimensions.
For even dimensions, we have

$$v(t,x) = \sum_{j=0}^{\frac{n-2}{2}} (j+1)b_j t^j (\partial_t^j G_1)(t,x) + t\sum_{j=0}^{\frac{n-2}{2}} b_j t^j \left((\partial_t^{j+1}G_1)(t,x) + (\partial_t^j G_2)(t,x)\right)$$

where $b_j = b_j(n)$ and G_1, G_2 are given by

$$\begin{aligned} G_1(t,x) &= \frac{1}{w_{n+1}} \int_{S^n} \phi(x_1+t\xi_1,\cdots,x_n+t\xi_n)\, d\xi, \\ G_2(t,x) &= \frac{1}{w_{n+1}} \int_{S^n} \psi(x_1+t\xi_1,\cdots,x_n+t\xi_n)\, d\xi \end{aligned}$$

or

$$G_1(t,x) = \frac{2\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})t^{n-1}}\int_0^t \frac{r^{n-1}}{w_n(t^2-r^2)^{1/2}}\int_{S^{n-1}}\phi(x+r\xi)\,d\xi dr,$$

$$G_2(t,x) = \frac{2\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})t^{n-1}}\int_0^t \frac{r^{n-1}}{w_n(t^2-r^2)^{1/2}}\int_{S^{n-1}}\psi(x+r\xi)\,d\xi dr$$

and we can do the similar calculation as we did for the odd dimensions. □

(4) Klein-Gordon equations

Consider the following Klein-Gordon equation:

$$u_{tt}-\Delta u+mu = 0, \quad u(t=0)=u_0,\ u_t(t=0)=u_1 \tag{2.9}$$

for $u_0 \in C_0^n(\mathbb{R}^n),\ u_1 \in C_0^{n-1}(\mathbb{R}^n)$ where $t \geqslant 0$ and $x \in \mathbb{R}^n,\ m>0$. And let u be a solution of (2.9). Then the following estimate holds.

Lemma 2.4 *There exists $C>0$ such that*

$$\|u(t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{n}{2}}(\|u_0\|_{W^{N_2,1}}+\|u_1\|_{W^{N_2-1,1}}) \tag{2.10}$$

where $N_2=n$ if $n \geqslant 2$ and $N_2=2$ if $n=1$.

Proof. Note that if we define $v := e^{-i\sqrt{m}x_{n+1}}u$ then $v_{tt}=e^{-i\sqrt{m}x_{n+1}}u_{tt}$ and $(\partial_1^2+\cdots\partial_{n+1}^2)v = e^{-i\sqrt{m}x_{n+1}}(\partial_1^2+\cdots\partial_n^2+m)u$ so that v solves

$$v_{tt}-\Delta v = e^{-i\sqrt{m}x_{n+1}}(u_{tt}-\Delta u+mu) = 0$$

with $v(0,x_1,\cdots,x_{n+1})=e^{-i\sqrt{m}x_{n+1}}u_0,\ v_t(0,x_1,\cdots,x_{n+1})=e^{-i\sqrt{m}x_{n+1}}u_1$. So we can apply the estimate of the wave equations above if the factor $e^{-i\sqrt{m}x_{n+1}}$ does not affect the estimates above. This is carried out in detail in Wahl [3]. □

3. Global existence of solutions to nonlinear dispersive wave equations

In this section, we consider the Cauchy problem for the nonlinear dispersive wave equation in 1 space dimension

$$\partial_t^2 u+\frac{1}{\rho^2}|\partial_x|^{2\rho}u=\lambda|\partial_t u|^{p-1}\partial_t u, \quad u(0,x)=u_0(x),\ \partial_t u(0,x)=u_1(x) \tag{3.1}$$

where $0<\rho\leqslant 2,\ \rho\neq 1,\ \lambda\in\mathbb{C},\ |\partial_x|^{2\rho}=(-\partial_x^2)^\rho=\mathcal{F}^{-1}|\xi|^{2\rho}\mathcal{F},\ (t,x)\in\mathbb{R}\times\mathbb{R}$. Note that

$$\frac{d}{dt}(\|\partial_t u(t)\|_{L^2}^2+\frac{1}{\rho^2}\||\partial_x|^\rho u(t)\|_{L^2}^2)=2\mathrm{Re}\lambda\|\partial_t u(t)\|_{L^{p+1}}^{p+1}.$$

We introduce new vector $w=(a|\partial_x|^\rho-i\rho b\partial_t)u$ where $a=\begin{bmatrix}1\\1\end{bmatrix}$ and $b=\begin{bmatrix}1\\-1\end{bmatrix}$. Note that $u=\frac{1}{2}|\partial_x|^{-\rho}a\cdot w$ and $\partial_t u=\frac{i}{2\rho}b\cdot w$. Define

$$\mathcal{L} = \begin{bmatrix}\mathcal{L}_+ & 0\\ 0 & \mathcal{L}_-\end{bmatrix} = \begin{bmatrix}\partial_t-\frac{i}{\rho}|\partial_x|^\rho & 0\\ 0 & \partial_t+\frac{i}{\rho}|\partial_x|^\rho\end{bmatrix}$$

and let the nonlinear term

$$\mathcal{N}(w) \;=\; 2^{-p} i\rho^{1-p}\lambda b|(b\cdot w)|^{p-1}(b\cdot w) \;=\; \begin{bmatrix} i\lambda\rho|\partial_t u|^{p-1}\partial_t u \\ -i\lambda\rho|\partial_t u|^{p-1}\partial_t u \end{bmatrix}.$$

Then the problem (3.1) can be written as a system of two nonlinear Schrödinger equations

$$\mathcal{L}w = \mathcal{N}(w),\;\; w(t=0) = w_0 \tag{3.2}$$

where $w_0 = a|\partial_x|^\rho u_0 - i\rho b u_1$. And define the free evolution group

$$\mathcal{U}(t) \;=\; \begin{bmatrix} \mathcal{W}(t) & 0 \\ 0 & \mathcal{W}(-t) \end{bmatrix}$$

where $\mathcal{W}(t) = e^{\frac{it}{\rho}|\partial_x|^\rho} = \mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}\mathcal{F}$. Applying $\mathcal{U}(-t)$ to (3.2) and using the identity $\mathcal{U}(-t)\mathcal{L} = \partial_t\mathcal{U}(-t)$ we get

$$\partial_t\mathcal{U}(-t)w \;=\; \mathcal{U}(-t)\mathcal{N}(w).$$

Integrating both sides respecto to t, we have

$$\mathcal{U}(-t)w \;=\; w_0 + \int_0^t \mathcal{U}(-\tau)\mathcal{N}(w)\,d\tau$$

so that

$$w = \mathcal{U}(t)w_0 + \int_0^t \mathcal{U}(t-\tau)\mathcal{N}(w)\,d\tau. \tag{3.3}$$

Note that

$$\|\partial_t u\|_{L^2}^2 + \frac{1}{\rho^2}\||\partial_x|^\rho u\|_{L^2}^2 = \frac{1}{4\rho^2}\|w\|_{L^2}^2.$$

Define the operators

$$\mathcal{J}_\pm \;=\; \mathcal{W}(\pm t)x\mathcal{W}(\mp t) = \mathcal{F}^{-1}e^{\pm\frac{it}{\rho}|\xi|^\rho} i\partial_\xi e^{\mp\frac{it}{\rho}|\xi|^\rho}\mathcal{F} = \mathcal{F}^{-1}(i\partial_\xi \pm t\xi|\xi|^{\rho-2})\mathcal{F} = x \mp it\partial_x|\partial_x|^{\rho-2}.$$

Then we have

$$[\mathcal{J}_\pm, \mathcal{L}_\pm] = 0,\; [\mathcal{P}, \mathcal{L}_\pm] = \rho\mathcal{L}_\pm,\; \mathcal{J}_\pm\partial_x = \mathcal{P} - \rho t\mathcal{L}_\pm$$

where $\mathcal{P} = x\partial_x + \rho t\partial_t,\; \mathcal{L}_\pm = \partial_t \mp \frac{i}{\rho}|\partial_x|^\rho$.

Lemma 3.1 *Let $2 \leqslant q < \infty$, $n < \frac{q}{q-1}$ and $\rho \in (0,1)\cup(1,2]$. Then the estimate*

$$\|u\|_{L^\infty} \;\leqslant\; Ct^{-\frac{1}{2}}\left(\|(-\Delta)^{\frac{s+\mu-1}{2}}u\|_{L^2} + \||x|(-\Delta)^{\frac{s+\mu}{2}}\mathcal{W}(\mp t)u\|_{L^2}\right)^{\frac{nq}{2n+2q-nq}}\|u\|_{H_0^{\mu,2}}^{1-\frac{nq}{2n+2q-nq}}$$

holds for all $t > 0$ provided that the right-hand side is finite, where $\mu = n - \frac{\rho}{2},\; s = \frac{n(q-2)}{2q}$.

Proof. By the Hölder's inequality choosing $\delta = \||x|f\|_{L^q}^{\frac{2q}{2n+2q-nq}}\|f\|_{L^2}^{-\frac{2q}{2n+2q-nq}} > 0$ we obtain

$$\begin{aligned}
\|f\|_{L^1} &= \int_{|x|\leqslant\delta}|f(x)|\,dx + \int_{|x|>\delta}|x||f(x)||x|^{-1}\,dx \\
&\leqslant \left(\int_{|x|\leqslant\delta}dx\right)^{1/2}\|f\|_{L^2} + \left(\int_{|x|>\delta}\frac{1}{|x|^{\frac{q}{q-1}}}\,dx\right)^{\frac{q-1}{q}}\||x|f\|_{L^q} \\
&\leqslant C\delta^{\frac{n}{2}}\|f\|_{L^2} + C\left(\int_\delta^\infty r^{n-1-\frac{q}{q-1}}\,dr\right)^{\frac{q-1}{q}}\||x|f\|_{L^q} \\
&\leqslant C\delta^{\frac{n}{2}}\|f\|_{L^2} + C\delta^{-\frac{n+q-nq}{q}}\||x|f\|_{L^q} \\
&= C\||x|f\|_{L^q}^{\frac{nq}{2n+2q-nq}}\|f\|_{L^2}^{1-\frac{nq}{2n+2q-nq}} + C\||x|f\|_{L^q}^{1-\frac{2n+2q-2nq}{2n+2q-nq}}\|f\|_{L^2}^{\frac{2n+2q-2nq}{2n+2q-nq}} \\
&\leqslant C\||x|f\|_{L^q}^{\frac{nq}{2n+2q-nq}}\|f\|_{L^2}^{1-\frac{nq}{2n+2q-nq}}
\end{aligned}$$

for $n < \frac{q}{q-1}$, $2 \leqslant q < \infty$. Then by the **Lemma 2.2**, we find

$$\begin{aligned}
\|u\|_{L^\infty} &= \|\mathcal{W}(\pm t)\mathcal{W}(\mp t)u\|_{L^\infty} \\
&\leqslant Ct^{-\frac{1}{2}}\|(-\Delta)^{\frac{\mu}{2}}\mathcal{W}(\mp t)u\|_{L^1} \\
&\leqslant Ct^{-\frac{1}{2}}\||x|(-\Delta)^{\frac{\mu}{2}}\mathcal{W}(\mp t)u\|_{L^q}^{\frac{nq}{2n+2q-nq}}\|(-\Delta)^{\frac{\mu}{2}}\mathcal{W}(\mp t)u\|_{L^2}^{1-\frac{nq}{2n+2q-nq}} \\
&\leqslant Ct^{-\frac{1}{2}}\||x|(-\Delta)^{\frac{\mu}{2}}\mathcal{W}(\mp t)u\|_{L^q}^{\frac{nq}{2n+2q-nq}}\|u\|_{H_0^{\mu,2}}^{1-\frac{nq}{2n+2q-nq}}.
\end{aligned}$$

Next we apply the Sobolev imbedding theorem

$$\|f\|_{L^q} \leqslant C\|(-\Delta)^{\frac{s}{2}}f\|_{L^2}$$

with $s = \frac{n(q-2)}{2q}$ (note that $s = \frac{n}{2}\cdot\frac{q-2}{q} < \frac{n}{2}$ if $2 \leqslant q < \infty$). Hence we have

$$\|u\|_{L^\infty} \leqslant Ct^{-\frac{1}{2}}\|(-\Delta)^{\frac{s}{2}}|x|(-\Delta)^{\frac{\mu}{2}}\mathcal{W}(\mp t)u\|_{L^2}^{\frac{nq}{2n+2q-nq}}\|u\|_{H_0^{\mu,2}}^{1-\frac{nq}{2n+2q-nq}}.$$

By the Plancherel's theorem we get

$$\begin{aligned}
\|(-\Delta)^{\frac{s}{2}}|x|(-\Delta)^{\frac{\mu}{2}}f\|_{L^2} &= \|\mathcal{F}(-\Delta)^{\frac{s}{2}}|x|(-\Delta)^{\frac{\mu}{2}}f\|_{L^2} = \||\xi|^s\partial_\xi|\xi|^\mu\mathcal{F}f\|_{L^2} \\
&\leqslant C\||\xi|^{s+\mu-1}\mathcal{F}f\|_{L^2} + C\|\partial_\xi|\xi|^{s+\mu}\mathcal{F}f\|_{L^2} \\
&= C\|(-\Delta)^{\frac{s+\mu-1}{2}}f\|_{L^2} + C\||x|(-\Delta)^{\frac{s+\mu}{2}}f\|_{L^2}.
\end{aligned}$$

Thus we finally get

$$\|u\|_{L^\infty} \leqslant Ct^{-\frac{1}{2}}\left(\|(-\Delta)^{\frac{s+\mu-1}{2}}u\|_{L^2} + \||x|(-\Delta)^{\frac{s+\mu}{2}}\mathcal{W}(\mp t)u\|_{L^2}\right)^{\frac{nq}{2n+2q-nq}}\|u\|_{H_0^{\mu,2}}^{1-\frac{nq}{2n+2q-nq}}$$

where $\mu = n - \frac{\rho}{2}$, $s = \frac{n(q-2)}{2q}$ for $2 \leqslant q < \infty$. This completes the proof of **Lemma 3.1**. □

Corollary 3.2 *Let $n = 1$, $\rho \in (0,1)\cup(1,2]$. Then the estimate*

$$\|u\|_{L^\infty} \leqslant Ct^{-\frac{1}{2}}\left(\||\partial_x|^{s+\mu-1}u\|_{L^2} + \|\mathcal{J}_\pm|\partial_x|^{s+\mu}u\|_{L^2}\right)^{\frac{q}{q+2}}\|u\|_{H^\mu}^{\frac{2}{q+2}}$$

holds for all $t > 0$ provided that the right-hand side is finite, where $\mu = 1 - \frac{\rho}{2}$, $s = \frac{q-2}{2q}$ for $2 \leqslant q < \infty$ (For the simplicity denote $H_0^{\mu,2} =: H^\mu$).

Proof. If $n = 1$ then $1 = n < \frac{q}{q-1}$ holds for all $2 \leqslant q < \infty$. And the exponent becomes $\frac{nq}{2n+2q-nq} = \frac{q}{2+q}$, $1 - \frac{nq}{2n+2q-nq} = \frac{2}{2+q}$ with $\mu = n - \frac{\rho}{2} = 1 - \frac{\rho}{2}$ and $s = \frac{n(q-2)}{2q} = \frac{q-2}{2q}$. □

Theorem 3.3 *Let $0 < \rho < 1$. Assume that*

$$\|x\partial_x w_0\|_{L^2} + \|w_0\|_{H^\mu} < \infty$$

and $p > 3$ where $\mu = 1 - \frac{\rho}{2}$. Then there exists an $\epsilon > 0$ such that the Cauchy problem (3.2) *has a unique global solution $w \in C([0,\infty), L^2)$ with the time decay estimate*

$$\|w(t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{1}{2}}$$

for any initial data w_0 satisfying $\|x\partial_x w_0\|_{L^2} + \|w_0\|_{H^\mu} \leqslant \epsilon$.

Proof.
We consider the linearized version of the integral equation (3.3)

$$w = \mathcal{U}(t)w_0 + \int_0^t \mathcal{U}(t-\tau)\mathcal{N}(v)\,d\tau \tag{3.4}$$

where $v \in X_a := \{v \in X : \|v\|_X \leqslant a\}$ where

$$X = \{v \in (C([0,\infty), L^2))^2 : \|v\|_X < \infty\},\ \ \|v\|_X = \sup_{t>0}(\|\mathcal{P}v\|_{L^2} + \|v\|_{H^\mu} + t\|\mathcal{L}v\|_{L^2})$$

with $\mu = 1 - \frac{\rho}{2},\ \mathcal{P} = \begin{bmatrix} x\partial_x + \rho t\partial_t & 0 \\ 0 & x\partial_x + \rho t\partial_t \end{bmatrix}$. Multiplying both sides of (3.4) by $\mathcal{P}$, we have

$$\mathcal{P}w \ \ = \ \ \mathcal{U}(t)(x\partial_x w_0) + \int_0^t \mathcal{U}(t-\tau)(\mathcal{P}+\rho)\mathcal{N}(v)\,d\tau$$

since $\mathcal{P}\mathcal{U}(t) = \mathcal{U}(t)x\partial_x$ (see **Appendix**). Taking L^2 norm both sides we have

$$\begin{aligned} \|\mathcal{P}w\|_{L^2} \ &\leqslant \ \|x\partial_x w_0\|_{L^2} + \int_0^t \|\mathcal{P}\mathcal{N}(v)\|_{L^2} + \rho\|\mathcal{N}(v)\|_{L^2}\,d\tau \\ &\leqslant \ \|x\partial_x w_0\|_{L^2} + C\int_0^t \|v(\tau)\|_{L^\infty}^{p-1}(\|\mathcal{P}v(\tau)\|_{L^2} + \|v(\tau)\|_{L^2})\,d\tau. \end{aligned}$$

By the estimate of **Corollary 3.2** with $q = \frac{2}{1-\rho} \geqslant 2$ (so that $s = \frac{q-2}{2q} = \frac{\rho}{2},\ s+\mu = \frac{\rho}{2}+1-\frac{\rho}{2} = 1,\ s+\mu-1 = 0,\ \frac{q}{q+2} = \frac{1}{2-\rho},\ \frac{2}{q+2} = \frac{1-\rho}{2-\rho}$), we get

$$\|v\|_{L^\infty} \ \leqslant \ Ct^{-\frac{1}{2}}(\|v\|_{L^2} + \|\mathcal{J}\partial_x v\|_{L^2})^{\frac{1}{2-\rho}}\|v\|_{H^\mu}^{\frac{1-\rho}{2-\rho}} \tag{3.5}$$

where

$$\mathcal{J} \ \ = \ \ \begin{bmatrix} \mathcal{J}_+ & 0 \\ 0 & J_- \end{bmatrix}.$$

Because $\mathcal{J}_\pm = (x\partial_x + \rho t\partial_t) - \rho t\mathcal{L}_\pm$, we have

$$\|\mathcal{J}\partial_x v\|_{L^2} \ \leqslant \ \|\mathcal{P}v\|_{L^2} + \rho t\|\mathcal{L}v\|_{L^2} \ \leqslant \ a + Ca.$$

Therefore (3.5) becomes

$$\|v\|_{L^\infty} \ \leqslant \ Ct^{-\frac{1}{2}}(a+a+Ca)^{\frac{1}{2-\rho}}a^{\frac{1-\rho}{2-\rho}} \ \leqslant \ Ct^{-\frac{1}{2}}a^{\frac{1}{2-\rho}}a^{\frac{1-\rho}{2-\rho}} \ \ = \ \ Cat^{-\frac{1}{2}}.$$

Also because $\mu = 1 - \frac{\rho}{2} > \frac{1}{2}$, by the Sobolev imbedding theorem, $\|v\|_{L^\infty} \leqslant C\|v\|_{H^\mu} \leqslant Ca$. Therefore we have the estimate

$$\|v\|_{L^\infty} \ \leqslant \ Ca(1+t)^{-\frac{1}{2}}. \tag{3.6}$$

Therefore we get for $p > 3$,

$$\begin{aligned} \|\mathcal{P}w\|_{L^2} \ &\leqslant \ \|x\partial_x w_0\|_{L^2} + C\int_0^t a^{p-1}(1+\tau)^{-\frac{p-1}{2}}(a+a)\,d\tau \\ &\leqslant \ \|x\partial_x w_0\|_{L^2} + Ca^p\int_0^t (1+\tau)^{-\frac{p-1}{2}}\,d\tau \ \leqslant \ \|x\partial_x w_0\|_{L^2} + Ca^p. \end{aligned}$$

Taking H^μ norm both sides of (3.4), similarly we get

$$\begin{aligned}
\|w\|_{H^\mu} &\leqslant \|w_0\|_{H^\mu} + \int_0^t \|\mathcal{N}(v)\|_{H^\mu}\, d\tau \\
&\leqslant \|w_0\|_{H^\mu} + \int_0^t \|v(\tau)\|_{L^\infty}^{p-1}\|v(\tau)\|_{H^\mu}\, d\tau \\
&\leqslant \|w_0\|_{H^\mu} + Ca^p \int_0^t (1+\tau)^{-\frac{p-1}{2}}\, d\tau \\
&\leqslant \|w_0\|_{H^\mu} + Ca^p.
\end{aligned}$$

Finally from $\mathcal{L}w = \mathcal{N}(v)$,

$$\|\mathcal{L}w\|_{L^2} \leqslant \|\mathcal{N}(v)\|_{L^2} \leqslant C\|v\|_{L^\infty}^{p-1}\|v\|_{L^2} \leqslant Ca^p(1+t)^{-\frac{p-1}{2}}.$$

Combining these results, we get

$$\begin{aligned}
\|w\|_X &= \sup_{t>0}(\|\mathcal{P}w\|_{L^2} + \|w\|_{H^\mu} + t\|\mathcal{L}w\|_{L^2}) \\
&\leqslant \|x\partial_x w_0\|_{L^2} + \|w_0\|_H^\mu + Ca^p + Ca^p \sup_{t>0} t(1+t)^{-\frac{p-1}{2}} \\
&\leqslant \|x\partial_x w_0\|_{L^2} + \|w_0\|_H^\mu + Ca^p \ \ (\because p > 3) \\
&\leqslant \epsilon + Ca^p
\end{aligned}$$

which implies that, if we take $a = \sqrt{\epsilon}$, then $\|w\|_X \leqslant \sqrt{\epsilon} = a.$ so that $w \in X_a$. In the same manner we can prove that the estimate for the difference. By the similar way, we have

$$\begin{aligned}
\|\mathcal{P}(w-\tilde{w})\|_{L^2} &\leqslant Ca^{p-1}\int_0^t (\|\mathcal{P}(v(\tau)-\tilde{v}(\tau))\|_{L^2} + \|v(\tau)-\tilde{v}(\tau)\|_{L^2})(1+\tau)^{-\frac{p-1}{2}}\, d\tau \\
&\leqslant Ca^{p-1}\int_0^t \sup_{\tau>0}(\|\mathcal{P}(v(\tau)-\tilde{v}(\tau))\|_{L^2} + \|v(\tau)-\tilde{v}(\tau)\|_{L^2})(1+\tau)^{-\frac{p-1}{2}}\, d\tau \\
&\leqslant Ca^{p-1}\|v-\tilde{v}\|_X \int_0^t (1+\tau)^{-\frac{p-1}{2}}\, d\tau \\
&\leqslant Ca^{p-1}\|v-\tilde{v}\|_X.
\end{aligned}$$

Also

$$\|w-\tilde{w}\|_{H^\mu} \leqslant Ca^{p-1}\|v-\tilde{v}\|_X$$

and

$$\|\mathcal{L}(w-\tilde{w})\|_{L^2} \leqslant Ca^{p-1}(1+t)^{-\frac{p-1}{2}}\|v-\tilde{v}\|_{L^2}.$$

Combining all these results, we get

$$\begin{aligned}
\|w-\tilde{w}\|_X &= \sup_{t>0}(\|\mathcal{P}(w-\tilde{w})\|_{L^2} + \|w-\tilde{w}\|_{H^\mu} + t\|\mathcal{L}(w-\tilde{w})\|_{L^2}) \\
&\leqslant Ca^{p-1}\|v-\tilde{v}\|_X
\end{aligned}$$

which implies that, if we take $a = (\frac{1}{2C})^{\frac{1}{p-1}}$, then

$$\|w-\tilde{w}\|_X \leqslant \frac{1}{2}\|v-\tilde{v}\|_X. \tag{3.7}$$

Therefore if we set the map T as (3.4):

$$T(\cdot) = \mathcal{U}(t)w_0 + \int_0^t \mathcal{U}(t-\tau)\mathcal{N}(\cdot)\, d\tau$$

then (3.7) becomes $\|Tv-T\tilde{v}\|_X \leqslant \frac{1}{2}\|v-\tilde{v}\|_X$ so that there exists unique w such that $Tw = w$ and $\|w\|_X \leqslant a$ by the contraction mapping principle, since X is a Banach space. Also note that (3.6) gives $\|w(t)\|_{L^\infty} \leqslant C(1+t)^{-\frac{1}{2}}$. □

Theorem 3.4 *Let $1 < \rho < 2$. Assume that*

$$\|(1+|\partial_x|^{-\beta})x\partial_x w_0\|_{L^2} + \||\partial_x|^{-\beta} w_0\|_{H^\sigma} < \infty$$

and $p > 2+\rho$ where $\frac{\rho-1}{2} < \beta < \min(\frac{1}{2}, \frac{p-3}{2})$, $\sigma > \frac{1}{2}+\beta$. Then there exists an $\epsilon > 0$ such that the Cauchy problem (3.2) has a unique global solution $w \in C([0,\infty), L^2)$ with the time decay estimate

$$\|w(t)\|_{L^\infty} \ \leqslant \ C(1+t)^{-\frac{1}{2}}$$

for any initial data w_0 satisfying $\|(1+|\partial_x|^{-\beta})x\partial_x w_0\|_{L^2} + \||\partial_x|^{-\beta} w_0\|_{H^\sigma} \leqslant \epsilon$.

Proof.
We consider (3.4) in the closed ball $Y_a = \{v \in Y : \|v\|_Y \leqslant a\}$ where

$$Y = \{v \in (C([0,\infty), L^2))^2 : \|v\|_Y < \infty\}$$

and the norm

$$\|v\|_Y = \sup_{t>0} \big(\|(1+|\partial_x|^{-\beta})\mathcal{P}v\|_{L^2} + \||\partial_x|^{-\beta} v\|_{H^\sigma} + t\||\partial_x|^{-\beta}\mathcal{L}v\|_{L^2}\big)$$

where $\beta = \frac{\rho-1}{2} + \gamma < \min(\frac{1}{2}, \frac{p-3}{2})$, $\sigma > \frac{1}{2}+\beta$ with small $\gamma > 0$.
Since we have $\mathcal{P}\mathcal{U}(t) = \mathcal{U}(t)x\partial_x$ and

$$\begin{aligned} |\partial_x|^{-\beta}\mathcal{U}(t) &= \mathcal{F}^{-1}|\xi|^{-\beta}\mathcal{F}\mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}\mathcal{F} = \mathcal{F}^{-1}|\xi|^{-\beta}e^{\frac{it}{\rho}|\xi|^\rho}\mathcal{F} \\ &= \mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}|\xi|^{-\beta}\mathcal{F} = \mathcal{F}^{-1}e^{\frac{it}{\rho}|\xi|^\rho}\mathcal{F}\mathcal{F}^{-1}|\xi|^{-\beta}\mathcal{F} = \mathcal{U}(t)|\partial_x|^{-\beta}, \end{aligned}$$

multiplying both sides of (3.4) by $(1+|\partial_x|^{-\beta})\mathcal{P}$ yields

$$(1+|\partial_x|^{-\beta})\mathcal{P}w \ = \ \mathcal{U}(t)(1+|\partial_x|^{-\beta})x\partial_x w_0 + \int_0^t \mathcal{U}(t-\tau)(1+|\partial_x|^{-\beta})(\mathcal{P}+\rho)\mathcal{N}(v)\,d\tau.$$

Therefore

$$\|(1+|\partial_x|^{-\beta})\mathcal{P}w\|_{L^2} \leqslant \|(1+|\partial_x|^{-\beta})x\partial_x w_0\|_{L^2} + \int_0^t \|(1+|\partial_x|^{-\beta})(\mathcal{P}+\rho)\mathcal{N}(v)\|_{L^2}\,d\tau. \quad (3.8)$$

By the Sobolev imbedding theorem $\||\partial_x|^{-\beta} f\|_{L^2} \leqslant C\|f\|_{L^q}$ ($\because \frac{1}{q} = \frac{1}{2}+\beta$), the generalized Hölder's inequality $\|f_1\cdots f_p\|_{L^q} \leqslant \|f_1\|_{L^{\frac{p-1}{\beta}}}\cdots\|f_{p-1}\|_{L^{\frac{p-1}{\beta}}}\|f_p\|_{L^2}$ ($\because \frac{1}{q} = \beta+\frac{1}{2} = \frac{\beta}{p-1}(p-1)+\frac{1}{2} = \frac{\beta}{p-1}+\cdots+\frac{\beta}{p-1}+\frac{1}{2}$), and the fact that $\|f\|_{L^p} \leqslant \|f\|_{L^\infty}^{1-\frac{2}{p}}\|f\|_{L^2}^{\frac{2}{p}}$ ($\because |f|^p \leqslant \|f\|_\infty^{p-2}|f|^2$), we obtain

$$\begin{aligned} &\big\|(1+|\partial_x|^{-\beta})(\mathcal{P}+\rho)\mathcal{N}(v)\big\|_{L^2} \\ &\quad\leqslant \ \||\partial_x|^{-\beta}(\mathcal{P}+\rho)\mathcal{N}(v)\|_{L^2} + \|(\mathcal{P}+\rho)\mathcal{N}(v)\|_{L^2} \\ &\quad\overset{\text{Sobolev}}{\leqslant} \ C\|(\mathcal{P}+\rho)\mathcal{N}(v)\|_{L^q} + \|(\mathcal{P}+\rho)\mathcal{N}(v)\|_{L^2} \\ &\quad\leqslant \ C\|\mathcal{P}\mathcal{N}(v)\|_{L^q} + C\|\mathcal{N}(v)\|_{L^q} + \|\mathcal{P}\mathcal{N}(v)\|_{L^2} + C\|\mathcal{N}(v)\|_{L^2} \\ &\quad\overset{\text{Hölder}}{\leqslant} \ C\|v\|_{L^{\frac{p-1}{\beta}}}^{p-1}\|\mathcal{P}v\|_{L^2} + C\|v\|_{L^{\frac{p-1}{\beta}}}^{p-1}\|v\|_{L^2} + \|v\|_{L^\infty}^{p-1}\|\mathcal{P}v\|_{L^2} + C\|v\|_{L^\infty}^{p-1}\|v\|_{L^2} \\ &\quad\leqslant \ C(\|v\|_{L^{\frac{p-1}{\beta}}}^{p-1} + \|v\|_{L^\infty}^{p-1})(\|\mathcal{P}v\|_{L^2} + \|v\|_{L^2}) \\ &\quad\leqslant \ C(\|v\|_{L^\infty}^{p-1-2\beta}\|v\|_{L^2}^{2\beta} + \|v\|_{L^\infty}^{p-1})(\|\mathcal{P}v\|_{L^2} + \|v\|_{L^2}). \end{aligned}$$

By the estimate of **Corollary 3.2** with sufficiently large $q = \frac{1}{\gamma}$ so that $\frac{\rho-1}{2} + \frac{1}{q} = \beta < \frac{1}{2}$ and $s + \mu - 1 = \frac{1}{2} - \frac{1}{q} - \frac{\rho}{2} = \frac{1-\rho}{2} - \gamma = -\beta$, we get

$$\begin{aligned} \|v\|_{L^\infty} &\leqslant Ct^{-\frac{1}{2}} \left(\||\partial_x|^{-\beta} v\|_{L^2} + \||\partial_x|^{-\beta} \mathcal{J}\partial_x v\|_{L^2} \right)^{\frac{q}{q+2}} \|v\|_{H^\mu}^{\frac{2}{q+2}} \\ &\leqslant Ct^{-\frac{1}{2}} (a + Ca)^{\frac{q}{q+2}} a^{\frac{2}{q+2}} \leqslant Cat^{-\frac{1}{2}} \end{aligned}$$

since

$$\||\partial_x|^{-\beta} \mathcal{J}\partial_x v\|_{L^2} \leqslant \||\partial_x|^{-\beta} \mathcal{P} v\|_{L^2} + \rho t \||\partial_x|^{-\beta} \mathcal{L} v\|_{L^2} \leqslant a + \rho a \leqslant Ca.$$

Also because $\sigma - \beta > \frac{1}{2}$, by the Sobolev imbedding theorem $\|v\|_{L^\infty} \leqslant C\||\partial_x|^{-\beta} v\|_{H^\sigma} \leqslant Ca$. Therefore we have the estimate

$$\|v\|_{L^\infty} \leqslant Ca(1+t)^{-\frac{1}{2}}. \tag{3.9}$$

Combining these results (3.8) becomes

$$\begin{aligned} \left\|(1 + |\partial_x|^{-\beta}) \mathcal{P} w(t)\right\|_{L^2} &\leqslant \|(1 + |\partial_x|^{-\beta}) x\partial_x w_0\|_{L^2} + Ca^p \int_0^t (1+\tau)^{\beta - \frac{p-1}{2}} + (1+\tau)^{-\frac{p-1}{2}} \, d\tau \\ &\leqslant \|(1 + |\partial_x|^{-\beta}) x\partial_x w_0\|_{L^2} + Ca^p \end{aligned}$$

since $p > 3 + 2\beta$ and $p > 2 + \rho > 3$. Similarly we have

$$\begin{aligned} \||\partial_x|^{-\beta} w\|_{H^\sigma} &\leqslant \||\partial_x|^{-\beta} w_0\|_{H^\sigma} + C \int_0^t (\|\mathcal{N}(v)\|_{L^q} + \||\partial_x|^{\sigma-\beta} \mathcal{N}(v)\|_{L^2}) \, d\tau \\ &\leqslant \||\partial_x|^{-\beta} w_0\|_{H^\sigma} + Ca^p \int_0^t (1+\tau)^{\beta - \frac{p-1}{2}} \, d\tau \\ &\leqslant \||\partial_x|^{-\beta} w_0\|_{H^\sigma} + Ca^p \end{aligned}$$

and

$$\||\partial_x|^{-\beta} \mathcal{L} w\|_{L^2} \leqslant \||\partial_x|^{-\beta} \mathcal{N}(v)\|_{L^2} \leqslant C\|v\|_{L^\infty}^{p-1-2\beta} \|v\|_{L^2}^{1+2\beta} \leqslant Ca^p (1+t)^{\beta - \frac{p-1}{2}}.$$

Combining all these results, we get

$$\begin{aligned} \|w\|_Y &= \sup_{t>0} \left(\|(1 + |\partial_x|^{-\beta}) \mathcal{P} w\|_{L^2} + \||\partial_x|^{-\beta} w\|_{H^\sigma} + t\||\partial_x|^{-\beta} \mathcal{L} w\|_{L^2} \right) \\ &\leqslant \|(1 + |\partial_x|^{-\beta}) x\partial_x w_0\|_{L^2} + \||\partial_x|^{-\beta} w_0\|_{H^\sigma} + Ca^p + Ca^p \sup_{t>0} t(1+t)^{\beta - \frac{p-1}{2}} \\ &\leqslant \|(1 + |\partial_x|^{-\beta}) x\partial_x w_0\|_{L^2} + \||\partial_x|^{-\beta} w_0\|_{H^\sigma} + Ca^p \\ &\leqslant \epsilon + Ca^p \end{aligned}$$

which implies that $\|w\|_Y \leqslant \sqrt{\epsilon}$ if we take $a = \sqrt{\epsilon}$. In the same manner, we prove the estimate for the difference

$$\|w - \tilde{w}\|_Y \leqslant \frac{1}{2} \|v - \tilde{v}\|_Y$$

by the similar way for the previous proof of the case $0 < \rho < 1$. Therefore (3.4) defines a contraction mapping in Y and (3.9) gives the time decay estimate. □

4. Appendix

(1) n-dimensional Spherical Coordinate System

We may define a coordinate system in an n-dimensional Euclidean space which is analogous to the spherical coordinate system defined for 3-dimensional Euclidean space, in which the coordinates consist of a radial coordinate, r and $n-1$ angular coordinates $\phi_1, \phi_2, \cdots, \phi_{n-1}$ where ϕ_{n-1} ranges over $[0, 2\pi)$, and the other angles range over $[0, \pi]$. If x_i are the Cartesian coordinates, then we may compute $x_1, \cdots, x_n$ from $r, \phi_1, \cdots, \phi_{n-1}$ with:

$$\begin{aligned}
x_1 &= r\cos(\phi_1) \\
x_2 &= r\sin(\phi_1)\cos(\phi_2) \\
x_3 &= r\sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \\
&\vdots \\
x_{n-1} &= r\sin(\phi_1)\cdots\sin(\phi_{n-2})\cos(\phi_{n-1}) \\
x_n &= r\sin(\phi_1)\cdots\sin(\phi_{n-2})\sin(\phi_{n-1})\,.
\end{aligned}$$

And the inverse transformation is:

$$\begin{aligned}
r &= \sqrt{{x_n}^2 + {x_{n-1}}^2 + \cdots + {x_2}^2 + {x_1}^2} \\
\phi_1 &= \operatorname{arccot}\frac{x_1}{\sqrt{{x_n}^2 + {x_{n-1}}^2 + \cdots + {x_2}^2}} \\
\phi_2 &= \operatorname{arccot}\frac{x_2}{\sqrt{{x_n}^2 + {x_{n-1}}^2 + \cdots + {x_3}^2}} \\
&\vdots \\
\phi_{n-2} &= \operatorname{arccot}\frac{x_{n-2}}{\sqrt{{x_n}^2 + {x_{n-1}}^2}} \\
\phi_{n-1} &= 2\operatorname{arccot}\frac{\sqrt{{x_n}^2 + {x_{n-1}}^2} + x_{n-1}}{x_n}.
\end{aligned}$$

And the Jacobian is given by

$$|J| \;=\; r^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\cdots\sin\phi_{n-2}$$

so that for our case, $n \geqslant 2$ then

$$\begin{aligned}
\int_{\mathbb{R}^n_+} f(|\zeta|)\,d\zeta &= \int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\int_0^{\infty} f(r)r^{n-1}\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}\,dr d\theta_1\cdots d\theta_{n-1} \\
&= \int_0^{\infty} f(r)r^{n-1}\,dr \times \int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}\,d\theta_1\cdots d\theta_{n-1} \\
&= \left(\prod_{k=0}^{n-2} W_k\right)\int_0^{\infty} f(r)r^{n-1}\,dr \;=\; C_n\int_0^{\infty} f(r)r^{n-1}\,dr
\end{aligned}$$

where

$$W_k \;=\; \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{1+k}{2}\right)}{\Gamma\left(1+\frac{k}{2}\right)}$$

which is given by the Wallis' integral. □

(2) $\mathcal{P}\mathcal{U}(t) = \mathcal{U}(t)x\partial_x$

To show this, we only have to prove that $(x\partial_x + \rho t\partial_t)\mathcal{W}(\pm t) = \mathcal{W}(\pm t)x\partial_x$. Because $\mathcal{F}xf = i\partial_\xi \mathcal{F}f$ and $\mathcal{F}\partial_x f = i\xi\mathcal{F}f$, we have

$$\begin{aligned} x\partial_x\mathcal{F}^{-1}e^{\pm\frac{it}{\rho}|\xi|^\rho}\mathcal{F} &= -\mathcal{F}^{-1}\partial_\xi\xi e^{\pm\frac{it}{\rho}|\xi|^\rho}\mathcal{F} \\ &= -\mathcal{F}^{-1}\left(e^{\pm\frac{it}{\rho}|\xi|^\rho}\mathcal{F} + \xi(\pm\frac{it}{\rho}\rho\xi|\xi|^{\rho-2}e^{\pm\frac{it}{\rho}|\xi|^\rho}\mathcal{F}) + \xi e^{\pm\frac{it}{\rho}|\xi|^\rho}\partial_\xi\mathcal{F}\right) \\ &= \mathcal{F}^{-1}\left(-e^{\pm\frac{it}{\rho}|\xi|^\rho} \mp it|\xi|^\rho e^{\pm\frac{it}{\rho}|\xi|^\rho} - \xi e^{\pm\frac{it}{\rho}|\xi|^\rho}\partial_\xi\right)\mathcal{F} \end{aligned}$$

and

$$\rho t\partial_t\mathcal{F}^{-1}e^{\pm\frac{it}{\rho}|\xi|^\rho}\mathcal{F} = \pm it\mathcal{F}^{-1}|\xi|^\rho e^{\pm\frac{it}{\rho}|\xi|^\rho}\mathcal{F},$$

we have

$$\begin{aligned} (x\partial_x + \rho t\partial_t)\mathcal{W}(\pm t) &= \mathcal{F}^{-1}\left(-e^{\pm\frac{it}{\rho}|\xi|^\rho} \mp it|\xi|^\rho e^{\pm\frac{it}{\rho}|\xi|^\rho} - \xi e^{\pm\frac{it}{\rho}|\xi|^\rho}\partial_\xi \pm it|\xi|^\rho e^{\pm\frac{it}{\rho}|\xi|^\rho}\right)\mathcal{F} \\ &= \mathcal{F}^{-1}\left(-e^{\pm\frac{it}{\rho}|\xi|^\rho} - \xi e^{\pm\frac{it}{\rho}|\xi|^\rho}\partial_\xi\right)\mathcal{F} \\ &= -\mathcal{F}^{-1}e^{\pm\frac{it}{\rho}|\xi|^\rho}\partial_\xi\xi\mathcal{F} \\ &= \mathcal{F}^{-1}e^{\pm\frac{it}{\rho}|\xi|^\rho}i\partial_\xi\mathcal{F}\partial_x \\ &= \mathcal{F}^{-1}e^{\pm\frac{it}{\rho}|\xi|^\rho}\mathcal{F}x\partial_x \\ &= \mathcal{W}(\pm t)x\partial_x \end{aligned}$$

which proves $\mathcal{P}\mathcal{U}(t) = \mathcal{U}(t)x\partial_x$. □

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