M1 Seminar II (Week 1)

Local Existence for Quasi-linear Symmetric Hyperbolic System

D. Kim

2012. 10. 25.

1. Preliminaries

We consider the following initial value problem

$$\partial_t^2 y - \Delta y = \sum_{i,j=1}^{n} a_{ij}(Dy)\partial_i\partial_j y \quad \text{with} \quad y(t=0) = y_0,\ \partial_t y(t=0) = y_1 \tag{1}$$

where $Dy := (\partial_t y, \partial_1 y, \cdots, \partial_n y)$. Also assume the following

$$a_{ij} = \overline{a_{ji}} \in C^\infty(\mathbb{R}\times\mathbb{R}^n),\ a_{ij}(0) = 0 \tag{2}$$

for all i, j. Assume (2) holds. Then we can show that there exists $m, M > 0$ and $\eta > 0$ such that for all $\xi \in \mathbb{C}^n$ and $v \in \mathbb{C}^{n+1}$, if $|v| < \eta$ then

$$m|\xi|^2 \leqslant |\xi|^2 + \sum_{i,j=1}^{n} a_{ij}(v)\xi_i\overline{\xi_j} \leqslant M|\xi|^2 \tag{3}$$

holds. Let A be the $n\times n$ matrix whose general term is $a_{ij}(v)$. Because a_{ij} is continuous and $a_{ij}(0) = 0$, there exists $\eta > 0$ such that if $|v| < \eta$ then $|a_{ij}(v)| < \frac{1}{2n}$ holds. By the Gershgorin circles theorem, the eigenvalues of $I + A(v)$ for such v are in the union of the circle of center $|1 + a_{ii}(v)|$ and radius $\frac{1}{2}$($\because R_i := \sum_{j\neq i}|a_{ij}| \leqslant \sum_{j=1}^{n}|a_{ij}| < \frac{1}{2}$). So for all the eigenvalue λ's, we have

$$|\lambda| \geqslant |1 + a_{ii}| - |\lambda - (1 + a_{ii})| \geqslant 1 - |a_{ii}| - \frac{1}{2} > 1 - \frac{1}{2n} - \frac{1}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right) := m > 0$$

$$|\lambda| \leqslant |1 + a_{ii}| + |\lambda - (1 + a_{ii})| \leqslant 1 + \frac{1}{2n} + \frac{1}{2} =: M > 1$$

respectively. Thus by considering the Rayleigh quotient R for $A + I$,

$$m \leqslant \lambda_{min} \leqslant R := \frac{\xi^*(I + A)\xi}{|\xi|^2} \leqslant \lambda_{max} \leqslant M$$

which proves (3). Moreover if A is positive definite, then we can eliminate $|\xi|^2$ term in (3) without the assumption $a_{ij}(0) = 0$. In this case, m, M are, in fact, λ_{min} and λ_{max} respectively.

Now consider more general situation:

$$A^0(u)\partial_t u + \sum_{j=1}^{n} A^j(u)\partial_j u + B(u)u = 0 \quad \text{with} \quad u(t=0) = u_0 \tag{4}$$

where $u = (u_1, \cdots, u_N) \in \mathbb{C}^N,\ u_j = u_j(t,x),\ t \in \mathbb{R},\ x \in \mathbb{R}^n$. Note that if we set $u := Dy,\ u_0 := (y_1, \nabla y_0)$, and

$$A^0 := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & b_{n1} & \cdots & b_{nn} \end{bmatrix},\ A^j := -\begin{bmatrix} 0 & b_{1j} & \cdots & b_{nj} \\ b_{1j} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_{nj} & 0 & \cdots & 0 \end{bmatrix},\ B := O$$

where $b_{ij} = b_{ij}(u) := a_{ij}(u) + \delta_{ij}$, then the initial value problem (4) is equivalent to (1). In (4), we assume that

$$\{A^j\}_{j=0}^n, B : N \times N \text{ matrix composed of } C^\infty \text{ functions} \tag{5}$$

$$\{A^j\}_{j=0}^n : \text{Hermitian},\ A^0 : \text{positive definite} \tag{6}$$

uniformly. So from now on we consider the initial value problem (4) with the assumptions (5), (6). First, we use some notations as below.

$$|u|_{s,T} := \sup_{t\in[0,T]} \|u(t)\|_{s,2} \ \text{ for } u \in L^\infty([0,T], W^{s,2}),\ T > 0,\ s \in \mathbb{N}_0,$$

$$\text{for } N \times N \text{ matrix } A,\ |A|^2 := \sum_{i,j=1}^N |a_{ij}|^2,\ \|A\|_p^p := \int_{\mathbb{R}^n} |A|^p,\ \|A\|_\infty := \sum_{i,j=1}^N \|a_{ij}\|_\infty.$$

2. Local Existence Theorem

Theorem 1. Assume (5) and (6) and let $u_0 \in W^{s,2},\ s \in \mathbb{N}$. And let $s > 1 + \frac{n}{2},\ g_1 := \kappa_s \|u_0\|_{s,2},\ g_2 > g_1$ (Here κ_s denotes the Sobolev constant characterizing the continuous embedding of $W^{s,2}$ into the space of bounded continuous functions, for example, if $s > n/2$, then $\|w\|_\infty \leqslant \kappa_s \|w\|_{s,2}$ for $w \in W^{s,2}$). Then there exists $T = T(\|u_0\|_{s,2}, g_2) > 0$ such that there exists a unique classical solution $u \in C_b^1([0,T] \times \mathbb{R}^n)$ of the initial value problem (4) with

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |u(t,x)| \leqslant g_2, \ \text{ and } \ u \in C^0([0,T], W^{s,2}) \cap C^1([0,T], W^{s-1,2}).$$

To prove the Theorem 1, we first prove the uniqueness of the solution and some lemmas for the purpose.

3. Uniqueness of the Solution

We will prove the uniqueness in the larger class $\mathscr{U}$ than that of Theorem 1. Define $\mathscr{U}$ as

$$\mathscr{U} := C^0([0,T], W^{1,2}) \cap C^1([0,T], L^2) \cap L^\infty([0,T], W^{s,2}).$$

Let $u_1, u_2 \in \mathscr{U}$ be solutions of (4). By putting u_1, u_2 into (4), if we define $v := u_2 - u_1$, then we have

$$\tilde{A}^0 \partial_t v + \sum_{j=1}^{n} \tilde{A}^j \partial_j v + \tilde{B} v = \tilde{F}, \quad v(t=0) = 0 \tag{7}$$

where $\tilde{A}^0 := A^0(u_2(t,x)),\ \tilde{A}^j := A^j(u_2(t,x)),\ \tilde{B} := B(u_2(t,x))$, and

$$\tilde{F} := (A^0(u_1) - A^0(u_2))\partial_t u_1 + \sum_{j=1}^{n} (A^j(u_1) - A^j(u_2))\partial_j u_1 + (B(u_1) - B(u_2))u_1.$$

Taking inner product both sides of (7) with $\overline{v}$, using the fact that $Re[X\partial_j v \cdot \overline{v}] = \frac{1}{2} Re[\partial_j (Xv \cdot \overline{v}) - (\partial_j X)v \cdot \overline{v})$($X$: Hermitian, $j \in \mathbb{N}_0$), and taking the integration over $\mathbb{R}^n$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \tilde{A}^0 v \cdot \overline{v} &= Re\left[\frac{1}{2}\int_{\mathbb{R}^n} (\partial_t \tilde{A}^0)v \cdot \overline{v} + \frac{1}{2}\int_{\mathbb{R}^n} \sum_{j=1}^{n} (\partial_j \tilde{A}^j)v \cdot \overline{v} - \int_{\mathbb{R}^n} \tilde{B} v \cdot \overline{v} + \int_{\mathbb{R}^n} \tilde{F} \cdot \overline{v}\right] \\
&\leqslant c\int_{\mathbb{R}^n} \left(|\partial_t A^0(u_2)||v|^2 + \sum_{j=1}^{n} |\partial_j A^j(u_2)||v|^2 + |B(u_2)||v|^2 + |\tilde{F}||v| \right) \\
&\leqslant c\int_{\mathbb{R}^n} \sum_{i,j=1}^{N} |\partial_t A^0_{ij}(u_2(t,x))||v|^2 + \cdots \\
&= c\int_{\mathbb{R}^n} \sum_{i,j=1}^{N} |\nabla_u A^0_{ij}(u_2) \cdot \partial_t u_2(t,x)||v|^2 + \cdots \\
&\leqslant c\int_{\mathbb{R}^n} \sum_{i,j=1}^{N} |\nabla_u A^0_{ij}(u_2)||\partial_t u_2||v|^2 + \cdots \\
&\leqslant c\int_{\mathbb{R}^n} \left(|\nabla_u A^0(u_2)||\partial_t u_2||v|^2 + \sum_{j=1}^{N} |\nabla_u A^j(u_2)||\partial_j u_2||v|^2 + |B(u_2)||v|^2 + |\tilde{F}||v| \right) \\
&\leqslant c\int_{\mathbb{R}^n} |v|^2 + c\int_{\mathbb{R}^n} |\tilde{F}||v| \qquad (8) \\
&\leqslant c\int_{\mathbb{R}^n} \tilde{A}^0 v \cdot \overline{v} + c\int_{\mathbb{R}^n} |\tilde{F}||v|. \qquad (9)
\end{aligned}$$

(8) : Using the Sobolev inequality and the fact that $u_k \in L^\infty([0,T], W^{s,2})$ for $k = 1, 2$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |u_k| = \sup_{t\in[0,T]} \|u_k\|_\infty \leqslant \kappa_s \sup_{t\in[0,T]} \|u_k\|_{s,2} < \infty$$

which means that u_1, u_2 are bounded, and so is $v = u_2 - u_1$. Because $\nabla_u A^0, \nabla_u A^j, B$ are composed of continuous functions, $\nabla_u A^0(u_k), \nabla_u A^j(u_k), B(u_k)$ are bounded. Because $s > 1 + n/2$, $\partial_j u_2$ is also bounded. And by the equation (4),

$$|\partial_t u_k| \leqslant c|A^0(u_k)^{-1}| \left(\sum_{j=1}^{n} |A^j(u_k)||\partial_j u_k| + |B(u_k)||u_k| \right) < \infty$$

because $(A^0)^{-1}$ is also positive definite and consists of continuous functions.

(9) : By the positive definiteness of $\tilde{A}^0$.

Now we can control $\tilde{F}$ as

$$\begin{aligned}
|\tilde{F}| &\leqslant c|A^0(u_1)-A^0(u_2)||\partial_t u_1| + c\sum_{j=1}^{n}|A^j(u_1)-A^j(u_2)||\partial_j u_1| + c|B(u_1)-B(u_2)||u_1| \\
&= c\left|\int_0^1 \frac{d}{dr}A^0(ru_1+(1-r)u_2)dr\right||\partial_t u_1| + \cdots \\
&\leqslant c\int_0^1 |\nabla_u A^0(ru_1+(1-r)u_2)|dr|u_1-u_2||\partial_t u_1| + \cdots \\
&\leqslant c\left(|\partial_t u_1| + \sum_{j=1}^{n}|\partial_j u_1| + |u_1|\right)|v| \\
&\leqslant c|v|.
\end{aligned}$$

Thus the inequality (9) becomes

$$\frac{d}{dt}\int_{\mathbb{R}^n}\tilde{A}^0 v\cdot\overline{v} \leqslant c\int_{\mathbb{R}^n}\tilde{A}^0 v\cdot\overline{v} + c\int_{\mathbb{R}^n}|v|^2 \leqslant c\int_{\mathbb{R}^n}\tilde{A}^0 v\cdot\overline{v} \tag{10}$$

by using the positive definiteness of $\tilde{A}^0$ again. Finally applying Gronwall's inequality to (10) with $v(t=0)=0$, for $t\in[0,T]$,

$$\int_{\mathbb{R}^n}\tilde{A}^0 v\cdot\overline{v} \leqslant \int_{\mathbb{R}^n}\tilde{A}^0 v\cdot\overline{v}(t=0)e^{ct} = 0$$

which implies that $u_1 = u_2$.

4. Existence of the Solution

Lemma 1. (*A Global Existence Theorem for the Linear Symmetric Hyperbolic System*) Let $L := A^0(t,x)\partial_t + \sum_{j=1}^n A^j(t,x)\partial_j + B(t,x)$, and assume the condition (5), (6) hold. Let $s\in\mathbb{N}, s > n/2+1$, let $A^0, A^1, \cdots, A^n, B \in C_b^\infty$ and let $u_0 \in W^{s,2}\cap C^\infty$. Then there exists a unique solution $u\in C^\infty([0,\infty)\times\mathbb{R}^n)$ to the initial value problem $Lu=0$ with $u(t=0)=u_0$. Moreover

$$u \in C^0([0,\infty), W^{s,2}) \cap C^1([0,\infty), W^{s-1,2}).$$

Now let $\epsilon_k := 2^{-k}\epsilon_0,\ k\in\mathbb{N}_0,\ 0<\epsilon_0\leqslant 1$. Define u_0^k by using Friedrichs mollifier, $u_0^k := J_{\epsilon_k}u_0 = j_{\epsilon_k} * u_0$. Then for any $k, s\in\mathbb{N}_0$, $u_0^k\in W^{s,2}\cap C^\infty$, $\|u_0^k\|_{s,2}\leqslant\|u_0\|_{s,2}$. Let $u^0(t,x) := u_0^0(x)$ for $x\in\mathbb{R}^n$ and fixed $t\geqslant 0$. Define $u^{k+1} = u^{k+1}(t,x)$ by iteration for $k\in\mathbb{N}_0$ as the solution of the linear symmetric hyperbolic system:

$$A^0(u^k)\partial_t u^{k+1} + \sum_{j=1}^n A^j(u^k)\partial_j u^{k+1} + B(u^k)u^{k+1} = 0,\ u^{k+1}(t=0) = u_0^{k+1}\ (k\in\mathbb{N}_0). \tag{11}$$

Observe $A^0, A^j, B \in C^\infty$, $u^0 = u_0^0 \in W^{s,2}\cap C^\infty$ so that $A^0(u^0), A^j(u^0), B(u^0)\in C_b^\infty$. So by the Lemma 1, there exists a unique global solution u^1 for the equation (11). Note that so there also exists a local solution u_{loc}^1 on $[0,M]$ for some large M. Then because $u_{loc}^1 \in C^0([0,M], W^{s,2})$, u_{loc}^1 is bounded so that all the coefficients are bounded in (11).

Therefore by the Lemma 1, again, there is a unique global solution u^2. By repeating these steps, the solution u^{k+1} will be well-defined with for all $k, m \in \mathbb{N}_0$,

$$u^{k+1} \in C^0([0,\infty), W^{s,2}) \cap C^1([0,\infty), W^{s-1,2}) \cap C^\infty([0,\infty)\times\mathbb{R}^n). \tag{12}$$

Our aim is that the sequence of solutions of (11), $\{u_k\}$, converges to the solution of (4). For this purpose, we will prove some lemmas.

Lemma 2. Let $N, m, n \in \mathbb{N}$, $A_{N\times N} \in C^m(\mathbb{C}^N)$, $B_M := \|A\|_{C^m(\overline{B(0,M)})}$. Then there exists $c = c(N,m,n) > 0$ such that for $u = (u_1, \cdots, u_N) \in W^{m,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\|u\|_\infty \leqslant M$,

$$\|\nabla^m A(u)\|_2 \leqslant NcB_M M^{m-1}\|\nabla^m u\|_2.$$

Here $\|\nabla^m A\|_2^2 := \sum_{|\alpha|=m} \|\nabla^\alpha A\|_2^2$ for the multi-index α.

Lemma 3. Let $N, m, n \in \mathbb{N}$. Then there exists $c = c(m,n) > 0$ such that for any $A_{N\times N}, u_{N\times 1} \in W^{m,2} \cap L^\infty$ and $\alpha \in \mathbb{N}_0^n$, $0 < |\alpha| \leqslant m$,

(i) $\|\nabla^\alpha(Au)\|_2 \leqslant c(\|A\|_\infty\|\nabla^m u\|_2 + \|\nabla^m A\|_2\|u\|_\infty)$
(ii) $\|\nabla^\alpha(Au) - A\nabla^\alpha u\|_2 \leqslant c(\|\nabla A\|_\infty\|\nabla^{m-1}u\|_2 + \|\nabla^m A\|_2\|u\|_\infty)$.

The proof of above lemmas have been done in the last seminars. Now we introduce a new lemma as below.

Lemma 4. (*Boundedness in High Norms*)
There exists $R,\ L,\ T_* > 0$ such that for any $k \in \mathbb{N}_0$,

(i) $|u^k|_{s,T_*} \leqslant R$ (ii) $|\partial_t u^k|_{s-1,T_*} \leqslant L$
(iii) For any $(t,x) \in [0,T_*]\times\mathbb{R}^n$, $|u^k(t,x)| \leqslant g_2$.

Here R, L, T_* are the function of $\|u_0\|_{s,2}, g_2$.

Proof of Lemma 4.
Note that $c_0(g_2), c_0^{-1}(g_2)$ denote for any $v, w \in \mathbb{C}^N$,

$$|w| \leqslant g_2 \ \Rightarrow\ c_0^{-1}(g_2)|v|^2 \leqslant A^0(w)v\cdot\overline{v} \leqslant c_0(g_2)|v|^2$$

and $c(g_2)$ denote various constants which depend on g_2 and on values of the coefficients $X(w)$ for $|w| \leqslant g_2$. The proof will be done by an induction. Our strategy is that (I) (i),(ii),(iii) hold for $k = 0$, (II) (i) implies (ii) and (ii) implies (iii) for all $k > 0$, and (III) (i) holds for $k+1$ when we assume (i) holds for k.

(I) $k = 0$**.**
(i) $|u^0|_{s,T_*} = \sup_{t\in[0,T_*]}\|u_0^0\|_{s,2} = \|u_0^0\|_{s,2} < \infty\ (\leqslant R < \infty)$
(ii) Because t was fixed for u^0, $\partial_t u^0(x) = 0$.
(iii) Because $u^0 \in W^{s,2}$, $|u^0| \leqslant \kappa_s\|u_0^0\|_{s,2} \leqslant \kappa_s\|u_0\|_{s,2} = g_1 < g_2$.

(II) (i) ⇒ (ii), (ii) ⇒ (iii).
(i) ⇒ (ii) : By (11), we have

$$\begin{aligned}\|\partial_t u^{k+1}\|_{s-1,2}^2 &= \left\| A^0(u^k)^{-1}\left(\sum_{j=1}^n A^j(u^k)\partial_j u^{k+1} + B(u^k)u^{k+1}\right)\right\|_{s-1,2}^2 \\ &\leqslant S_1 + S_2 + S_3 \end{aligned} \tag{13}$$

where

$$S_1 := \left\| A^0(u^k)^{-1}\left(\sum_{j=1}^n A^j(u^k)\partial_j u^{k+1} + B(u^k)u^{k+1}\right)\right\|_2^2,$$

$$S_2 := \sum_{1\leqslant|\alpha|\leqslant s-1}\left\|\nabla^\alpha\left(A^0(u^k)^{-1}\sum_{j=1}^n A^j(u^k)\partial_j u^{k+1}\right)\right\|_2^2,$$

$$S_3 := \sum_{1\leqslant|\alpha|\leqslant s-1}\left\|\nabla^\alpha\left(A^0(u^k)^{-1}B(u^k)u^{k+1}\right)\right\|_2^2.$$

Now reduce S_1 by using (i),

$$\begin{aligned} S_1 &\leqslant c\left\|\sum_{j=1}^n A^j(u^k)\partial_j u^{k+1} + B(u^k)u^{k+1}\right\|_2^2 \\ &\leqslant c\left(\sum_{j=1}^n \|\partial_j u^{k+1}\|_2^2 + \|u^{k+1}\|_2^2\right) = c\|u^{k+1}\|_{1,2}^2 \leqslant cR^2. \end{aligned} \tag{14}$$

For S_2, we will use Lemma 2 and Lemma 3 (with $m = |\alpha|$).

$$\begin{aligned} S_2 &\leqslant \sum_{1\leqslant m\leqslant s-1} c\left(\|A^0(u^k)^{-1}\|_\infty^2\left\|\nabla^m\left(\sum_{j=1}^n A^j(u^k)\partial_j u^{k+1}\right)\right\|_2^2 + \|\nabla^m A^0(u^k)^{-1}\|_2^2\left\|\sum_{j=1}^n A^j(u^k)\partial_j u^{k+1}\right\|_\infty^2\right) \\ &\leqslant c\sum_{1\leqslant m\leqslant s-1}\left(\sum_{j=1}^n \|\nabla^m(A^j(u^k)\partial_j u^{k+1})\|_2^2 + c(g_2)\|\nabla^m u^k\|_2^2\sum_{j=1}^n c\|\partial_j u^{k+1}\|_\infty^2\right) \\ &\leqslant c(g_2)\sum_{1\leqslant m\leqslant s-1}\sum_{j=1}^n\left(c\|A^j(u^k)\|_\infty^2\|\nabla^m\partial_j u^{k+1}\|_2^2 + c\|\nabla^m A^j(u^k)\|_2^2\|\partial_j u^{k+1}\|_\infty^2 + \|\nabla^m u^k\|_2^2\|\partial_j u^{k+1}\|_\infty^2\right) \\ &\leqslant c(g_2)\sum_{1\leqslant m\leqslant s-1}\sum_{j=1}^n\left(\|\nabla^m\partial_j u^{k+1}\|_2^2 + c(g_2)\|\nabla^m u^k\|_2^2\|\partial_j u^{k+1}\|_\infty^2 + \|\nabla^m u^k\|_2^2\|\partial_j u^{k+1}\|_\infty^2\right) \\ &\leqslant c(g_2)\sum_{1\leqslant m\leqslant s-1}\sum_{j=1}^n(\|\nabla^m\partial_j u^{k+1}\|_2^2 + \|\nabla^m u^k\|_2^2\|\partial_j u^{k+1}\|_\infty^2) \\ &\leqslant c(g_2)\left(\sum_{1\leqslant|\alpha|\leqslant s-1}\|\nabla^{\alpha+1}u^{k+1}\|_2^2 + \left(\sum_{1\leqslant|\alpha|\leqslant s-1}\|\nabla^\alpha u^k\|_2^2\right)c\kappa_s^2\|u^{k+1}\|_{s,2}^2\right) \\ &\leqslant c(g_2)(\|u^{k+1}\|_{s,2}^2 + c\|u^k\|_{s,2}^2\|u^{k+1}\|_{s,2}^2) \\ &\leqslant c(g_2)(R^2 + R^4). \end{aligned}$$

By the same way, we have also $S_3 \leqslant c(g_2)(R^2+R^4)$. Therefore the inequality (13) becomes

$$\|\partial_t u^{k+1}\|^2_{s-1,2} \leqslant c(g_2)(R^2+2R^2+2R^4) \leqslant c(g_2)R^2(1+R^2)$$

so that

$$|\partial_t u^{k+1}|_{s-1,T_*} \leqslant c(g_2)R\sqrt{1+R^2} =: L(R,g_2) = L > 0.$$

(ii) $\Rightarrow$ (iii) : Assume (ii), then

$$\begin{aligned} |u^{k+1}(t,x)| &\leqslant |u^{k+1}(0,x)| + \int_0^t |\partial_t u^{k+1}(r,x)|dr \\ &\leqslant \kappa_s \|u_0\|_{s,2} + \int_0^t \kappa_{s-1}\|\partial_t u^{k+1}(r)\|_{s-1,2}dr \\ &= g_1 + \kappa_{s-1}LT_* \leqslant g_2 \end{aligned}$$

where

$$0 < T_* \leqslant T_1 = T_1(\|u_0\|_{s,2}, g_2, R) := \frac{g_2 - g_1}{\kappa_{s-1}L}. \tag{15}$$

(III) Proof of (i) (Induction step). For simplicity, we write $u := u^k$, $v := u^{k+1}$, $v_0 := u_0^{k+1}$. By the differential equations, we have

$$\partial_t v = -A^0(u)^{-1}\left(\sum_{j=1}^n A^j(u)\partial_j v + B(u)v\right). \tag{16}$$

Take ∇^α both sides of (16) and multiply by $A^0(u)$, then

$$A^0(u)\partial_t \nabla^\alpha v + \sum_{j=1}^n A^j(u)\partial_j \nabla^\alpha v + B(u)\nabla^\alpha v = F_\alpha \tag{17}$$

where

$$F_\alpha := A^0(u)\left[\sum_{j=1}^n \left((A^0)^{-1}A^j(u)\partial_j\nabla^\alpha v - \nabla^\alpha((A^0)^{-1}A^j\partial_j v)\right) + (A^0)^{-1}B(u)\nabla^\alpha v - \nabla^\alpha((A^0)^{-1}B(u)v)\right] \tag{18}$$

Taking the inner product with $\overline{\nabla^\alpha v}$, using the fact that $Re(X\partial_j v\cdot\overline{v}) = \frac{1}{2}Re(\partial_j(Xv\cdot\overline{v}) - (\partial_j X)v\cdot\overline{v})$, integrating over $\mathbb{R}^n$, and summing up for $0 \leqslant |\alpha| \leqslant s$ of the both sides of (18),

$$\frac{d}{dt}\sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n} A^0(u)\nabla^\alpha v\cdot\overline{\nabla^\alpha v} = Re\left[\sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n}(D\cdot A(u) - 2B(u))\nabla^\alpha v\cdot\overline{\nabla^\alpha v} + 2\sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n} F_\alpha\cdot\overline{\nabla^\alpha v}\right] \tag{19}$$

where $D := [\partial_t \ \partial_1 \ \cdots \ \partial_n]^T$, $A := [A^0 \ A^1 \ \cdots \ A^n]^T$.

Now observe that the induction hypothesis implies for any $t \in [0, T_*]$,

$$\|u(t)\|_\infty \leqslant g_2 \ \text{ (by (iii))}$$

$$\|\nabla u(t)\|_\infty \leqslant \kappa_s |u|_{s,T_*} \leqslant \kappa_s R \ \text{(by (i))}$$

and

$$\begin{aligned}
\|\partial_t u(t)\|_\infty &= \left\| A^0(u^{k-1})^{-1} \left(\sum_{j=1}^n A^j(u^{k-1}) \partial_j u^k + B(u^{k-1}) u^k \right) \right\|_\infty \\
&\leqslant c(g_2) \left\| \sum_{j=1}^n \partial_j u(t) + u(t) \right\|_\infty \\
&\leqslant c(g_2) \|u(t)\|_{1,\infty} \leqslant c(g_2) \|u(t)\|_{s,2} \leqslant c(g_2) R \ \text{(by (i))}.
\end{aligned}$$

Thus for any $(t,x) \in [0,T_*] \times \mathbb{R}^n$,

$$\begin{aligned}
|D \cdot A(u(t,x))| &= \left| \partial_t A^0(u) + \sum_{j=1}^n \partial_j A^j(u) \right| \\
&\leqslant c \left(|\nabla_u A^0(u)||\partial_t u| + \sum_{j=1}^n |\nabla_u A^j(u)||\partial_j u| \right) \\
&\leqslant c(g_2) R
\end{aligned} \tag{20}$$

and

$$|B(u(t,x))| \leqslant c(g_2). \tag{21}$$

And by (ii) of the Lemma 3 with $m = |\alpha| > 0$ ($\alpha = 0 \Rightarrow F_\alpha = 0$),

$$\begin{aligned}
\|F_\alpha\|_2 &\leqslant c \sum_{j=1}^n \left(\|\nabla((A^0)^{-1} A^j(u))\|_\infty \|\nabla^{m-1} \partial_j v\|_2 + \|\nabla^m((A^0)^{-1} A^j(u))\|_2 \|\partial_j v\|_\infty \right) \\
&+ c(\|\nabla((A^0)^{-1} B(u)\|_\infty \|\nabla^{m-1} v\|_2 + \|\nabla^m((A^0)^{-1} B(u))\|_2 \|v\|_\infty) \\
&\leqslant \cdots \\
&\leqslant c(g_2) R \|v\|_{s,2}.
\end{aligned} \tag{22}$$

So by using the inequality (20), (21), (22), the equation (19) becomes

$$\begin{aligned}
\frac{d}{dt} \sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^n} A^0(u) \nabla^\alpha v \cdot \overline{\nabla^\alpha v} &\leqslant c(g_2)(R+1) \|v\|_{s,2}^2 + c \sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^n} |F_\alpha| |\nabla^\alpha v| \\
&\leqslant c(g_2)(R+1) \|v\|_{s,2}^2 + c \sum_{|\alpha| \leqslant s} \|F_\alpha\|_2 \|\nabla^\alpha v\|_2 \\
&\leqslant c(g_2)(R+1) \|v\|_{s,2}^2 + c(g_2) R \|v\|_{s,2}^2 \\
&\leqslant c(g_2)(R+1) \|v\|_{s,2}^2.
\end{aligned} \tag{23}$$

Integrating both sides of (23) from 0 to t yields

$$\begin{aligned}
\sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^n} A^0(u) \nabla^\alpha v \cdot \overline{\nabla^\alpha v} &\leqslant \sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^n} A^0(u_0) \nabla^\alpha v_0 \cdot \overline{\nabla^\alpha v_0} + c(g_2)(R+1) \int_0^t \|v(r)\|_{s,2}^2 dr \\
&\leqslant c_0(g_2) \|v_0\|_{s,2}^2 + c(g_2)(R+1) \int_0^t \|v(r)\|_{s,2}^2 dr
\end{aligned}$$

so that

$$\|v(t)\|_{s,2}^2 \leqslant c_0(g_2) \|v_0\|_{s,2}^2 + c_0(g_2) c(g_2)(R+1) \int_0^t \|v(r)\|_{s,2}^2 dr. \tag{24}$$

Applying the Gronwall's inequality to (24) gives

$$\|v(t)\|_{s,2} \leqslant c_0(g_2)\|u_0^{k+1}\|_{s,2}e^{c(g_2)(R+1)t} \leqslant c_0(g_2)\|u_0\|_{s,2}e^{c(g_2)(R+1)T_*}. \tag{25}$$

So here we choose $R \geqslant c_0(g_2)\|u_0\|_{s,2}e^{c(g_2)}$ and $T_* \leqslant \frac{1}{R+1} =: T_2$ so that $\|v(t)\|_{s,2} \leqslant R$. Finally combine the conditions (i) of (I) and (15) with these conditions:

$$R := \max\{\|u_0\|_{s,2},\ c_0(g_2)\|u_0\|_{s,2}e^{c(g_2)}\}, \quad T_* := \min\{T_1, T_2\}.$$

Then $R,\ L,\ T_*$ are the function of $\|u_0\|_{s,2},\ g_2$ and we have $|u^{k+1}|_{s,T_*} = \sup_{t\in[0,T_*]}\|v\|_{s,2} \leqslant R$ which proves the Lemma 4.

References.

[1] R. Racke. *Lectures on Nonlinear Evolution Equations.* Vieweg, 1991.
[2] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables.* Appl. Math. Sci. 53. Springer-Verlag, 1984.