M1 Seminar II (Week 2)

Local Existence for Quasi-linear Symmetric Hyperbolic System

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2012. 11. 8.

1. Preliminaries

Last week, we considered the quasi-linear symmetric hyperbolic system:

$$A^0(u)\partial_t u + \sum_{j=1}^{n} A^j(u)\partial_j u + B(u)u = 0 \quad \text{with} \quad u(t=0) = u_0 \tag{1}$$

where $u = (u_1, \cdots, u_N) \in \mathbb{C}^N,\ u_j = u_j(t,x),\ t \in [0,\infty),\ x \in \mathbb{R}^n$.

We derived the sequence of functions u^{k+1} by the solution of the linear symmetric hyperbolic system:

$$A^0(u^k)\partial_t u^{k+1} + \sum_{j=1}^{n} A^j(u^k)\partial_j u^{k+1} + B(u^k)u^{k+1} = 0 \quad \text{with} \quad u^{k+1}(t=0) = u_0^{k+1}(:= J_{\epsilon_{k+1}} u_0) \tag{2}$$

iteratively. Moreover for any $k, s \in \mathbb{N}_0$,

$$u^{k+1} \in C^0([0,\infty), W^{s,2}) \cap C^1([0,\infty), W^{s-1,2}) \cap C^\infty([0,\infty)\times\mathbb{R}^n). \tag{3}$$

Also we have proved the following lemma.

2. Lemmas

Lemma 1. There exists $R, L, T_* > 0$ such that for any $k \in \mathbb{N}_0$,

(i) $|u^k|_{s,T_*} \leqslant R$ (ii) $|\partial_t u^k|_{s-1,T_*} \leqslant L$ (iii) $\forall (t,x) \in [0,T_*]\times\mathbb{R}^n,\ |u^k(t,x)| \leqslant g_2$.

And the following elementary lemma holds.

Lemma 2.

$$\forall \epsilon_0 > 0,\ \exists c > 0\ s.t.\ \forall \epsilon \in (0,\epsilon_0],\ \forall u \in W^{1,2},\ \|J_\epsilon u - u\|_2 \leqslant c\epsilon\|u\|_{1,2}.$$

3. Theorem

Theorem 1. *(Convergence in Low Norms)*
There exists $T, \alpha, \beta_1, \beta_2, \cdots \geqslant 0$ with $0 < \alpha < 1$, $\sum_{k=1}^{\infty} \beta_k < \infty$, $0 < T \leqslant T_*$ such that for any $k \in \mathbb{N}$,

$$|u^{k+1} - u^k|_{0,T} \leqslant \alpha |u^k - u^{k-1}|_{0,T} + \beta_k.$$

Proof.
Let $0 \leqslant t \leqslant T \leqslant T_*$. By the partial differential equation for u^{k+1}, u^k,

$$\begin{aligned} &A^0(u^k)\partial_t(u^{k+1} - u^k) + \sum_{j=1}^{n} A^j(u^k)\partial_j(u^{k+1} - u^k) + B(u^k)(u^{k+1} - u^k) \\ &= (A^0(u^{k-1}) - A^0(u^k))\partial_t u^k + \sum_{j=1}^{n}(A^j(u^{k-1}) - A^j(u^k))\partial_j u^k + (B(u^{k-1}) - B(u^k))u^k =: F_k \end{aligned} \tag{4}$$

By the almost same way as before, we can control F_k as

$$\begin{aligned} |F_k| &\leqslant c|A^0(u^{k-1}) - A^0(u^k)||\partial_t u^k| + \cdots \\ &= c\left|\int_0^1 \frac{d}{dr} A^0(u^{k-1} + r(u^k - u^{k-1}))dr\right| |\partial_t u^k| + \cdots \\ &\leqslant c\int_0^1 |\nabla_u A^0(u^{k-1} + r(u^k - u^{k-1}))|dr|u^k - u^{k-1}||\partial_t u^k| + \cdots \\ &\leqslant c(g_2)\left(|\partial_t u^k| + \sum_{j=1}^{n} |\partial_j u^k| + |u^k|\right)|u^k - u^{k-1}| \\ &\leqslant c(g_2, R)|u^k - u^{k-1}| \end{aligned}$$

so that $\|F_k\|_2 \leqslant c(g_2, R)\|u^k - u^{k-1}\|_2$. Taking the supremum for $t \in [0, T]$,

$$|F_k|_{0,T} \leqslant c(g_2, R)|u^k - u^{k-1}|_{0,T}.$$

For the simplicity let $v^k := u^{k+1} - u^k$. Then by the symmetry, (4) becomes

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} A^0(u^k)v^k \cdot \overline{v^k} \\ &= \frac{1}{2}\int_{\mathbb{R}^n} (\partial_t A^0(u^k))v^k \cdot \overline{v^k} + \frac{1}{2}\int_{\mathbb{R}^n} \left(\sum_{j=1}^{n} \partial_j A^j(u^k)\right) v^k \cdot \overline{v^k} - Re\int_{\mathbb{R}^n} B(u^k)v^k \cdot \overline{v^k} + Re\int_{\mathbb{R}^n} F_k \cdot \overline{v^k} \\ &\leqslant c(g_2, R)\left(\int_{\mathbb{R}^n} |v^k|^2 + \|F_k\|_2\|v^k\|_2\right) \\ &\leqslant c(g_2, R)(\|v^k\|_2^2 + |v^{k-1}|_{0,T}\|v^k\|_2) \\ &\leqslant c(g_2, R)(\|v^k\|_2^2 + |v^{k-1}|_{0,T}^2). \end{aligned}$$

Integrating both sides from 0 to t,

$$\int_{\mathbb{R}^n} A^0(u^k)v^k \cdot \overline{v^k} \leqslant c(g_2, R)\left(A^0(u_0^k)v_0^k \cdot \overline{v_0^k} + T|v^{k-1}|_{0,T}^2 + \int_0^t \|v^k\|_2^2\right).$$

By the positive definiteness of A^0,

$$\|v^k(t)\|_2^2 \leqslant c(g_2, R)\left(\|v_0^k\|_2^2 + T|v^{k-1}|_{0,T}^2 + \int_0^t \|v^k(r)\|_2^2 dr\right).$$

So by the Gronwall's inequality,

$$\|v^k(t)\|_2 \leqslant \sqrt{c(g_2, R)}(\|v_0^k\|_2 + \sqrt{T}|v^{k-1}|_{0,T})e^{c(g_2,R)T}.$$

Taking supremum over $t \in [0, T]$ both sides we have

$$|u^{k+1} - u^k|_{0,T} \leqslant \sqrt{c(g_2, R)}e^{c(g_2,R)T}(\|u_0^{k+1} - u_0^k\|_2 + \sqrt{T}|u^k - u^{k-1}|_{0,T}) \tag{5}$$

Now by Lemma 2,

$$\begin{aligned} \|u_0^{k+1} - u_0^k\|_2 &\leqslant \|u_0^{k+1} - u_0\|_2 + \|u_0 - u_0^k\|_2 = \|J_{\epsilon_{k+1}}u_0 - u_0\|_2 + \|J_{\epsilon_k}u_0 - u_0\|_2 \\ &\leqslant c\epsilon_0 2^{-(k+1)}\|u_0\|_{1,2} + c\epsilon_0 2^{-k}\|u_0\|_{1,2} \leqslant c\|u_0\|_{1,2}2^{-k}. \end{aligned}$$

Now choose $0 < T \leqslant T_*$ such that $\alpha := \sqrt{c(g_2, R)T}e^{c(g_2,R)T} < 1$ holds. Let $\beta_k := c2^{-k}$, then (5) becomes

$$|u^{k+1} - u^k|_{0,T} \leqslant \alpha|u^k - u^{k-1}|_{0,T} + \beta_k$$

for all k and the conditions of α, β_k are satisfied.

4. Corollaries

Corollary 1. There exists $u \in C^0([0, T], L^2)$ such that

$$u^k \longrightarrow u \text{ in } C^0([0, T], L^2).$$

Proof. Let $0 \leqslant a_{k+1} \leqslant \alpha a_k + \beta_k,\ 0 \leqslant \alpha < 1, \beta_k \geqslant 0, \sum_{k=1}^{\infty} \beta_k < \infty$. Then

$$s_N = \sum_{k=1}^{N} a_k \leqslant a_1 + \sum_{k=1}^{N}(\alpha a_k + \beta_k) \leqslant a_1 + \alpha s_N + \sum_{k=1}^{\infty} \beta_k$$

so that

$$s_N \leqslant \frac{1}{1-\alpha}\left(a_1 + \sum_{k=1}^{\infty} \beta_k\right) < \infty.$$

Thus $\sum_{k=1}^{\infty} a_k < \infty$ which implies that $\lim_{k\to\infty} a_k = 0$.
Now let $a_k := |u^{k+1} - u^k|_{0,T}$ then u^k is Cauchy in $C^0([0,T], L^2)$ which is Banach space. Thus $u^k \longrightarrow u$ in $C^0([0,T], L^2)$.

Corollary 2. For $s' \in \mathbb{N}_0$ with $0 \leqslant s' < s$, $u \in C^0([0,T], W^{s',2})$ and

$$u^k \longrightarrow u \text{ in } C^0([0,T], W^{s',2}).$$

Proof. By (i) of Lemma 1, for any $k \in \mathbb{N}_0$, $|u^k|_{s,T} \leqslant R$. Then Gagliardo-Nirenberg's inequality implies that

$$\begin{aligned} |u^k - u^m|_{s',T} &\leqslant c|u^k - u^m|_{0,T}^{1-s'/s}|u^k - u^m|_{s,T}^{s'/s} \\ &\leqslant c|u^k - u^m|_{0,T}^{1-s'/s} \ (k,m \in \mathbb{N}_0,\ s' \in \{0,1,\cdots,s\}) \\ &\longrightarrow 0 \ (k,m \to \infty,\ 0 \leqslant s' < s) \end{aligned}$$

which means that u^k is Cauchy in the Banach space $C^0([0,T], W^{s',2})$ for $0 \leqslant s' < s$.

Corollary 3. $u \in C^1([0,T], L^2)$ and u satisfies (1).

Because $s - 1 > n/2, u \in C^0([0,T], W^{s-1,2})$ and

$$u^k \longrightarrow u \text{ in } C^0([0,T], C_b^0).$$

Thus

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |u(t,x)| = \lim_{k\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |u^k(t,x)| \leqslant g_2.$$

Now for the convenience, let $A^0(\cdot)^{-1}\sum_{j=1}^n A^j(\cdot) =: \mathscr{A}(\cdot)$, $A^0(\cdot)^{-1}B(\cdot) =: \mathscr{B}(\cdot)$. Then by the partial differential equation (2), we have

$$\begin{aligned} &\left| -\partial_t u^{k+1} - A^0(u)^{-1}\left(\sum_{j=1}^n A^j(u)\partial_j u + B(u)u\right)\right|_{s-2,T} \\ &= \left|\mathscr{A}(u^k)\partial_j u^{k+1} + \mathscr{B}(u^k)u^{k+1} - \mathscr{A}(u)\partial_j u - \mathscr{B}(u)u\right|_{s-2,T} \\ &\leqslant \left|\mathscr{A}(u^k)\partial_j u^{k+1} + \mathscr{B}(u^k)u^{k+1} - \mathscr{A}(u^k)\partial_j u - \mathscr{B}(u^k)u\right|_{s-2,T} \\ &\qquad + \left|\mathscr{A}(u^k)\partial_j u + \mathscr{B}(u^k)u - \mathscr{A}(u)\partial_j u - \mathscr{B}(u)u\right|_{s-2,T} \\ &= \left|\mathscr{A}(u^k)(\partial_j u^{k+1} - \partial_j u) + \mathscr{B}(u^k)(u^{k+1} - u)\right|_{s-2,T} + \left|(\mathscr{A}(u^k) - \mathscr{A}(u))\partial_j u + (\mathscr{B}(u^k) - \mathscr{B}(u))u\right|_{s-2,T} \\ &\leqslant c(g_2)\left(\sum_{j=1}^n |\partial_j u^{k+1} - \partial_j u|_{s-2,T} + |u^{k+1} - u|_{s-2,T}\right) \\ &\qquad + c\left(\sum_{j=1}^n |\partial_j u|_{s-2,T}|\mathscr{A}(u^k) - \mathscr{A}(u)|_{s-2,T} + |u|_{s-2,T}|\mathscr{B}(u^k) - \mathscr{B}(u)|_{s-2,T}\right) \longrightarrow 0 \ (k \to \infty) \end{aligned}$$

The last limit ($\longrightarrow 0$) holds by the Corollary 2, continuity of $\mathscr{A}, \mathscr{B}$, and the boundedness of $|\partial_j u|_{s-2,T}, |u|_{s-2,T}$ by the Sobolev embedding theorem.
Thus $\partial_t u^{k+1}$ converges to $-A^0(u)^{-1}(\sum_{j=1}^n A^j(u)\partial_j u + B(u)u)$ in $C^0([0,T], W^{s-2,2})$ which is continuously embedded in $C^0([0,T], L^2)$ so that $u \in C^1([0,T], L^2)$ and u satisfies (1).

References.

[1] R. Racke. *Lectures on Nonlinear Evolution Equations.* Vieweg, 1991.
[2] H. Ringstrom. *The Cauchy Problem in General Relativity.* European Mathematical Society, 2009.