M1 Seminar II (Week 3)

Regularities of the Solution

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1. Preliminaries

We have proved the followings until now:

Lemma 1.

(i) $|u^k|_{s,T} \leqslant R$ (ii) $|\partial_t u^k|_{s-1,T} \leqslant L$ (iii) $\forall (t,x) \in [0,T] \times \mathbb{R}^n,\ |u^k(t,x)| \leqslant g_2$.

Lemma 2.

(i) There exists $u \in C^0([0,T], L^2)$ such that $u^k \longrightarrow u$ in $C^0([0,T], L^2)$.
(ii) $u \in C^0([0,T], W^{s',2})$ and $u^k \longrightarrow u$ in $C^0([0,T], W^{s',2})$ for $s' < s$.
(iii) $u \in C^0([0,T], C_b^0)$ with $\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |u(t,x)| \leqslant g_2$.
(iv) $u \in C^1([0,T], L^2)$ and u solves the PDE.

2. Theorems

Theorem 1. $u \in L^\infty([0,T], W^{s,2})$ and $|u|_{s,T} \leqslant R$.

Proof. By (i) of **Lemma 1**, for any $t \in [0,T],\ k \in \mathbb{N}_0,\ \|u^k(t)\|_{s,2} \leqslant R$. Note that $W^{s,2}$ is Hilbert space with the inner product which is defined by

$$\langle u, v \rangle_s := \sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^n} \nabla^\alpha u \cdot \overline{\nabla^\alpha v}$$

and Hilbert space is reflexive, by the **Alaoglu's theorem**, there exists a subsequence $u^{k'}(t)$ of $u^k(t)$ and $w_t \in W^{s,2}$ such that

$$u^{k'}(t) \rightharpoonup w_t \quad \text{in} \quad W^{s,2} \quad (k' \to \infty)$$

where the $\rightharpoonup$ means the weak convergence, i.e.,

$$\forall \psi \in W^{s,2},\ \langle u^{k'}(t), \psi \rangle_s \longrightarrow \langle w_t, \psi \rangle_s \quad (k' \to \infty).$$

Also note that $\|w_t\|_{s,2} \leqslant R$ because

$$\|w_t\|_{s,2}^2 \leqslant \limsup |\langle u^{k'}(t), w_t \rangle_s| \leqslant \limsup \|u^{k'}(t)\|_{s,2} \|w_t\|_{s,2} \leqslant R \|w_t\|_{s,2}.$$

Now our aim is that w_t, is in fact $u(t)$. For this purpose define the functional F_h for $h \in L^2$ as

$$F_h : W^{s,2} \ni g \mapsto \int_{\mathbb{R}^n} g\overline{h} =: F_h g \in \mathbb{C}.$$

Then because F_h is linear and bounded on $W^{s,2}$, by the **Riesz representation theorem**, for any $h \in L^2$, there exists $\psi = \psi(h) \in W^{s,2}$ such that $F_h g = \langle g, \psi \rangle_s$ for all $g \in W^{s,2}$. Thus for any $h \in L^2$, we have

$$\int_{\mathbb{R}^n} w_t \overline{h} = F_h w_t = \langle w_t, \psi \rangle_s = \lim_{k' \to \infty} \langle u^{k'}(t), \psi \rangle_s = \lim_{k' \to \infty} F_h u^{k'}(t) = \lim_{k' \to \infty} \int_{\mathbb{R}^n} u^{k'}(t)\overline{h} = \int_{\mathbb{R}^n} u(t)\overline{h}.$$

Note that the last equality holds because by the **Lemma 2**, $u^{k'} \longrightarrow u$ in $C^0([0,T], L^2)$ so that

$$\left| \int_{\mathbb{R}^n} u^{k'}\overline{h} - \int_{\mathbb{R}^n} u\overline{h} \right| \leqslant \|u^{k'} - u\|_2 \|h\|_2 \longrightarrow 0 \quad (k' \to \infty).$$

Therefore we have $w_t = u(t)$ and $\|u(t)\|_{s,2} = \|w_t\|_{s,2} \leqslant R$ which proves the theorem. □

Theorem 2. (i) $u \in C_w([0,T], W^{s,2})$, and (ii) $u \in Lip([0,T], W^{s-1,2})$.

Here $C_w([0,T], E)$ means the space of weakly continuous functions from $[0,T]$ into E, and $Lip([0,T], E)$ means the space of Lipschitz continuous functions from $[0,T]$ into E. Therefore $u \in C_w([0,T], W^{s,2})$ means that for any sequence $\{t_n\} \subset [0,T]$ satisfying $\lim_{n\to\infty} t_n = t_0$, $u(t_n)$ converges weakly to $u(t_0)$ in $W^{s,2}$.

Proof.
(i) Let $\{t_n\} \subset [0,T]$ such that $\lim_{n\to\infty} t_n = t_0$. Note that because $u \in C^0([0,T], L^2)$, $u(t_n) \longrightarrow u(t_0)$ in L^2. By the **Theorem 1**, we have $\|u(t_n)\|_{s,2} \leqslant R$. Thus by the **Alaoglu's theorem** again, there exists a subsequence $\{t_{n'}\}$ and $w_t \in W^{s,2}$ such that $u(t_{n'}) \rightharpoonup w_t$ in $W^{s,2}$. Define the functional F_h as before. Then the **Riesz representation theorem** implies that for any $h \in L^2$, there exists $\psi \in W^{s,2}$ such that $F_h g = \langle g, \psi \rangle_s$ for any $g \in W^{s,2}$. Thus for all $h \in L^2$,

$$\int_{\mathbb{R}^n} w_t \overline{h} = F_h w_t = \langle w_t, \psi \rangle_s = \lim_{n' \to \infty} \langle u(t_{n'}), \psi \rangle_s = \lim_{n' \to \infty} F_h u(t_{n'}) = \lim_{n' \to \infty} \int_{\mathbb{R}^n} u(t_{n'})\overline{h} = \int_{\mathbb{R}^n} u(t_0)\overline{h}.$$

Thus $w_t = u(t_0)$ and $u(t_{n'}) \rightharpoonup u(t_0)$ in $W^{s,2}$. But all subsequences weakly converge to the same one, we may conclude that $u(t_n) \rightharpoonup u(t_0)$ in $W^{s,2}$ which proves (i).

(ii) Let $0 \leqslant t_1 \leqslant t_2 \leqslant T$. Then by the **Lemma 1**, we have

$$\|u^k(t_2) - u^k(t_1)\|_{s-1,2} = \left\| \int_{t_1}^{t_2} \partial_t u^k(t) dt \right\|_{s-1,2} \leqslant \int_{t_1}^{t_2} \|\partial_t u^k(t)\|_{s-1,2} dt \leqslant L|t_2 - t_1|.$$

Then by the **Lemma 2**, $u^k \longrightarrow u$ in $C^0([0,T], W^{s-1,2})$, we have $\|u(t_2) - u(t_1)\|_{s-1,2} \leqslant L|t_2 - t_1|$. □

Now we will show that $u \in C^0([0,T], W^{s,2}) \cap C^1([0,T], W^{s-1,2})$ so that $u \in C_b^1([0,T] \times \mathbb{R}^n)$ by the Sobolev embedding theorem. For this purpose observe 3 things as below.

Observation 1. $u \in C^0([0,T], W^{s,2})$ implies $u \in C^1([0,T], W^{s-1,2})$.

Let $t_k \subset [0,T]$ such that $\lim_{k\to\infty} t_k = t_0$. Then by the PDE,

$$
\begin{aligned}
&\|\partial_t u(t_k) - \partial_t u(t_0)\|_{s-1,2} \\
&= \|\mathscr{A}(u(t_k))\partial_j u(t_k) + \mathscr{B}(u(t_k))u(t_k) - \mathscr{A}(u(t_0))\partial_j u(t_0) - \mathscr{B}(u(t_0))u(t_0)\|_{s-1,2} \\
&\leqslant \|\mathscr{A}(u(t_k))\partial_j u(t_k) + \mathscr{B}(u(t_k))u(t_k) - \mathscr{A}(u(t_k))\partial_j u(t_0) - \mathscr{B}(u(t_k))u(t_0)\|_{s-1,2} \\
&\qquad + \|\mathscr{A}(u(t_k))\partial_j u(t_0) + \mathscr{B}(u(t_k))u(t_0) - \mathscr{A}(u(t_0))\partial_j u(t_0) - \mathscr{B}(u(t_0))u(t_0)\|_{s-1,2} \\
&\leqslant c(g_2)(\|u(t_k) - u(t_0)\|_{s,2} + \|u(t_k) - u(t_0)\|_{s-1,2}) \\
&\qquad + c(\|\mathscr{A}(u(t_k)) - \mathscr{A}(u(t_0))\|_{s-1,2}\|u(t_0)\|_{s,2} + \|\mathscr{B}(u(t_k)) - \mathscr{B}(u(t_0))\|_{s-1,2}\|u(t_0)\|_{s-1,2}) \\
&\longrightarrow 0 \quad \text{as} \quad k \to \infty.
\end{aligned}
$$

The last limit holds because of the boundedness of $\|u(t_0)\|_{s,2}, \|u(t_0)\|_{s-1,2}$ by the Sobolev embedding theorem, and the continuity of u and $\mathscr{A}, \mathscr{B}$. Therefore $\partial_t u \in C^0([0,T], W^{s-1,2})$ so that $u \in C^1([0,T], W^{s-1,2})$.

Observation 2. The right continuity on $[0,T)$ implies the left continuity on $(0,T]$.

If we define $v(t) := u(T-t)$ for $0 \leqslant t < T$, the left continuity of $u(t)$ at $t_0 \in (0,T]$ is equivalent to the right continuity of $v(t)$ at $T - t_0 \in [0,T)$. And v satisfies

$$
A^0(v)\partial_t v + \sum_{j=1}^{n}(-A^j(v))\partial_j v + (-B(v))v = 0 \quad \text{with} \quad v(t=0) = u(t=T) \in W^{s,2}.
$$

Note that A^0 is positive definite and $-A^j, -B$ are again Hermitian, so we can do the same things until now for this new initial value problem. Thus the right continuity of $v(t)$ at $T - t_0 \in [0,T)$ is equivalent to the right continuity of $u(t)$ at $t_0 \in [0,T)$.

Observation 3. The right continuity at $t = 0$ implies the right continuity on $[0,T)$.

Let $t_0 \in (0,T)$. Consider the initial value problem

$$
A^0(w)\partial_t w + \sum_{j=1}^{n} A^j(w)\partial_j w + B(w)w = 0 \quad \text{with} \quad w(t=t_0) = u(t_0) \in W^{s,2}.
$$

Then by the same way, there exists a local solution w of this system on $[t_0, T_{t_0}] =: I$ and w satisfies

$$
w \in C^0(I, W^{s-1,2}) \cap C^1(I, L^2) \cap L^\infty(I, W^{s,2}) \cap C_w(I, W^{s,2}) \cap Lip(I, W^{s-1,2}).
$$

But we already proved the uniqueness in the space $\mathscr{U} := C^0([0,T], W^{1,2}) \cap C^1([0,T], L^2) \cap L^\infty([0,T], W^{s,2})$. So $w \in \mathscr{U}$ for $t \in I \cap [0,T]$. Thus $w = u$ on $[t_0, \min\{T_{t_0}, T\}]$. So the right continuity of u at $t_0 \in (0,T)$ is equivalent to the right continuity of w at t_0 which is the initial time of w. And this is also equivalent to the right continuity of u at the initial time which is $t = 0$. Thus the right continuity of u at $t = 0$ implies the right continuity of u at $t_0 \in (0,T)$.

So we only have to show the right continuity of u at $t = 0$. For this purpose define

$$\|v\|^2_{s,A^0(t)} := \sum_{|\alpha|\leqslant s} \int_{\mathbb{R}^n} A^0(t)\nabla^\alpha v \cdot \overline{\nabla^\alpha v}, \quad \text{where} \quad A^0(t) := A^0(u(t,\cdot))$$

for $t \in [0,T]$.
Note that there exists $c_0^{-1}, c_0 > 0$ such that

$$c_0^{-1}\|v\|^2_{s,2} \leqslant \|v\|^2_{s,A^0(t)} \leqslant c_0\|v\|^2_{s,2}$$

for $t \in [0,T]$ by the positive definiteness of A^0 and the upper bound for A^0.

Now observe that

$$\begin{aligned}
&\left|\|u(t)\|^2_{s,A^0(t)} - \|u(t)\|^2_{s,A^0(0)}\right| \\
&\qquad = \left|\sum_{|\alpha|\leqslant s} \int_{\mathbb{R}^n} (A^0(t) - A^0(0))\nabla^\alpha u \cdot \overline{\nabla^\alpha u}\right| \\
&\qquad \leqslant c \sum_{|\alpha|\leqslant s} \sup_{x\in\mathbb{R}^n} |A^0(u(t,x)) - A^0(u(0,x))| \|\nabla^\alpha u(t)\|_2^2 \\
&\qquad = c \sum_{|\alpha|\leqslant s} \sup_{x\in\mathbb{R}^n} \left|\int_0^1 \frac{d}{dr} A^0(u_0(x) + r(u(t,x) - u_0(x)))dr\right| \|\nabla^\alpha u(t)\|_2^2 \\
&\qquad \leqslant c \sup_{x\in\mathbb{R}^n} \int_0^1 \left|(\nabla_u A^0)(u_0(x) + r(u(t,x) - u_0(x)))\right| dr |u(t,x) - u_0(x)| \|u(t)\|^2_{s,2} \\
&\qquad \leqslant c(g_2)\|u(t) - u_0\|_{s-1,2} R^2.
\end{aligned}$$

The last inequality holds because of $\sup_{t,x} |u| \leqslant g_2$, the Sobolev embedding theorem and the **Theorem 1**. Now by taking limit superior both sides, we have

$$\limsup_{t\downarrow 0} \|u(t)\|^2_{s,A^0(t)} = \limsup_{t\downarrow 0} \|u(t)\|^2_{s,A^0(0)}.$$

Note that for the Hilbert space $\mathcal{H}$, if $w_n \rightharpoonup w$ in $\mathcal{H}$, then

$$w_n \longrightarrow w \ \text{ in } \ \mathcal{H} \qquad \Longleftrightarrow \qquad \|w\|_{\mathcal{H}} \geqslant \limsup \|w_n\|_{\mathcal{H}}.$$

Now we have $u \in C_w([0,T], W^{s,2})$, so by using this fact, we now only have to show that

$$\|u_0\|^2_{s,A^0(0)} \geqslant \limsup_{t\downarrow 0} \|u(t)\|^2_{s,A^0(t)}.$$

References.

[1] R. Racke. *Lectures on Nonlinear Evolution Equations.* Vieweg, 1991.
[2] J. B. Conway. *A Course in Functional Analysis.* Springer, 1990.