M1 Seminar II (Week 4)

Regularities of the Solution and an Improved Existence Theorem

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2012. 11. 22.

1. Preliminaries

Note 1. $(W^{s,2}, \langle\cdot,\cdot\rangle_{s,A^0(t)})$ is a Hilbert space with the inner product given by

$$\langle u, v\rangle_{s,A^0(t)} := \sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n}(A^0(t)\nabla^\alpha u)\cdot\overline{\nabla^\alpha v}$$

for $t\in[0,T]$ and a positive definite matrix A^0. Note that this is an inner product. The other properties of the inner product trivially hold except the symmetry, that is,

$$\begin{aligned}\overline{\langle u,v\rangle} &= \sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n}\overline{A^0\nabla^\alpha u\cdot\overline{\nabla^\alpha v}} = \sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n}\overline{A^0\nabla^\alpha u}\cdot\nabla^\alpha v\\ &= \sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n}\sum_{i,j=1}^N\overline{a_{ij}}\,\overline{\nabla^\alpha u_j}\nabla^\alpha v_i = \sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n}\sum_{i,j=1}^N a_{ji}\overline{\nabla^\alpha u_j}\nabla^\alpha v_i\\ &= \sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n}\sum_{i,j=1}^N a_{ij}\nabla^\alpha v_j\overline{\nabla^\alpha u_j} = \sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n}A^0\nabla^\alpha v\cdot\overline{\nabla^\alpha u} = \langle v,u\rangle.\end{aligned}$$

And the norm which is induced by this inner product $\|\cdot\|_{s,A^0(t)}$ is equivalent to the norm which is induced by the general inner product in the Sobolev space; $\|\cdot\|_{s,2}$. □

Note 2. Let H be a Hilbert space and let $w_n \rightharpoonup w$ in H. Then

$$w_n\longrightarrow w \text{ in } H \iff \|w\|_H\geqslant\limsup\|w_n\|_H$$

holds. Because

$$\|w\|^2=\langle w,w\rangle=\lim\langle w,w_n\rangle\leqslant\liminf\|w\|\|w_n\|$$

holds so that $\|w\|\leqslant\liminf\|w_n\|$. So the right hand side is equivalent to $\lim\|w_n\|=\|w\|$. And because $\|w_n-w\|^2=\|w_n\|^2-2Re\langle w_n,w\rangle+\|w\|^2$, $\lim\|w_n\|=\|w\|$ implies $\lim\|w_n-w\|=0$. The reverse is trivial by the triangular inequality. □

So because $u\in C_w([0,T],W^{s,2})$, it is now sufficient to prove that

$$\|u_0\|^2_{s,A^0(0)}\geqslant\limsup_{t\downarrow0}\|u(t)\|^2_{s,A^0(0)}.$$

If this holds then $u(t) \longrightarrow u_0$ as $t \downarrow 0$ which implies $u \in C^0([0,T], W^{s,2})$ so that $u \in C^1([0,T], W^{s,2})$ and $u \in C_b^1([0,T] \times \mathbb{R}^n)$.

2. Proof of the Regularity

To prove the regularity, first we will show that

$$\limsup_{t \downarrow 0} \|u(t)\|^2_{s,A^0(0)} = \limsup_{t \downarrow 0} \|u(t)\|^2_{s,A^0(t)}. \tag{1}$$

We have

$$\begin{aligned}
&\left| \|u(t)\|^2_{s,A^0(t)} - \|u(t)\|^2_{s,A^0(0)} \right| \\
&\qquad = \left| \sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^n} (A^0(t) - A^0(0)) \nabla^\alpha u(t) \cdot \overline{\nabla^\alpha u(t)} \right| \\
&\qquad \leqslant c \sum_{|\alpha| \leqslant s} \sup_{x \in \mathbb{R}^n} \left| A^0(u(t,x)) - A^0(u_0(x)) \right| \|\nabla^\alpha u(t)\|_2^2 \\
&\qquad = c \sum_{|\alpha| \leqslant s} \sup_{x \in \mathbb{R}^n} \left| \int_0^1 \frac{d}{dr} A^0(u_0(x) + r(u(t,x) - u_0(x)))dr \right| \|\nabla^\alpha u(t)\|_2^2 \\
&\qquad \leqslant c \sup_{x \in \mathbb{R}^n} \int_0^1 \left| \nabla_u A^0(u_0(x) + r(u(t,x) - u_0(x))) \right| dr |u(t,x) - u_0(x)| \|u(t)\|^2_{s,2} \\
&\qquad \leqslant c(g_2) \|u(t) - u_0\|_{s-1,2} R^2.
\end{aligned}$$

The last inequality holds by the fact $\sup_{t,x} |u| \leqslant g_2$, Sobolev embedding theorem, and the previous lemma. Because $u(t)$ converges to u_0 in $W^{s-1,2}$, taking the limit superior both sides yields (1). So we are now enough to show that

$$\|u_0\|^2_{s,A^0(0)} \geqslant \limsup_{t \downarrow 0} \|u(t)\|^2_{s,A^0(t)}. \tag{2}$$

Lemma 1. There exists $c = c(g_2, R)$ such that for any $t \in [0,T]$,

$$\|u(t)\|^2_{s,A^0(t)} \leqslant \|u_0\|^2_{s,A^0(0)} + ct.$$

Proof. From the proof of the previous lemma, using the fact $\|u^{k+1}\|_{s,2} \leqslant R$,

$$\frac{d}{dt} \sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^n} A^0(u^k) \nabla^\alpha u^{k+1} \cdot \overline{\nabla^\alpha u^{k+1}} \leqslant c(g_2)(R+1) \|u^{k+1}\|^2_{s,2} \leqslant c(g_2, R) =: c.$$

Integrating both sides from 0 to t,

$$\sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^n} A^0(u^k) \nabla^\alpha u^{k+1} \cdot \overline{\nabla^\alpha u^{k+1}} \leqslant \sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^n} A^0(u_0^k) \nabla^\alpha u_0^{k+1} \cdot \overline{\nabla^\alpha u_0^{k+1}} + ct. \tag{3}$$

On the other hand,

$$\begin{aligned}
&\left|\sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n} A^0(u_0^k)\nabla^\alpha u_0^{k+1}\cdot\overline{\nabla^\alpha u_0^{k+1}} - \sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n} A^0(u_0)\nabla^\alpha u_0\cdot\overline{\nabla^\alpha u_0}\right| \\
&\leqslant \int_{\mathbb{R}^n}\sum_{|\alpha|\leqslant s}\left|(A^0(u_0^k)-A^0(u_0))\nabla^\alpha u_0\cdot\overline{\nabla^\alpha u_0}\right| + \int_{\mathbb{R}^n}\sum_{|\alpha|\leqslant s}\left|A^0(u_0^k)(\nabla^\alpha u_0^{k+1}-\nabla^\alpha u_0)\cdot\overline{\nabla^\alpha u_0^{k+1}}\right| \\
&\qquad + \int_{\mathbb{R}^n}\sum_{|\alpha|\leqslant s}\left|A^0(u_0^k)\nabla^\alpha u_0\cdot(\nabla^\alpha u_0^{k+1}-\nabla^\alpha u_0)\right| \\
&\leqslant c\int_{\mathbb{R}^n}|A^0(u_0^k)-A^0(u_0)|\sum_{|\alpha|\leqslant s}|\nabla^\alpha u_0|^2 + c\int_{\mathbb{R}^n}|A^0(u_0^k)|\sum_{|\alpha|\leqslant s}|\nabla^\alpha u_0^{k+1}-\nabla^\alpha u_0||\nabla^\alpha u_0^{k+1}| \\
&\qquad + c\int_{\mathbb{R}^n}|A^0(u_0^k)|\sum_{|\alpha|\leqslant s}|\nabla^\alpha u_0||\nabla^\alpha u_0^{k+1}-\nabla^\alpha u_0| \\
&\leqslant c\sup_x|A^0(u_0^k)-A^0(u_0)|\|u_0\|_{s,2}^2 + c\sup_x|A^0(u_0^k)|\sum_{|\alpha|\leqslant s}\|\nabla^\alpha u_0^{k+1}-\nabla^\alpha u_0\|_2\|\nabla^\alpha u_0^{k+1}\|_2 \\
&\qquad + c\sup_x|A^0(u_0^k)|\sum_{|\alpha|\leqslant s}\|\nabla^\alpha u_0\|_2\|\nabla^\alpha u_0^{k+1}-\nabla^\alpha u_0\|_2 \\
&\leqslant A+B
\end{aligned}$$

where

$$A = c\sup_x|A^0(u_0^k)-A^0(u_0)|\|u_0\|_{s,2}^2,$$

$$B = c\sup_x|A^0(u_0^k)|\|u_0^{k+1}-u_0\|_{s,2}\|u_0^{k+1}\|_{s,2} + c\sup_x|A^0(u_0^k)|\|u_0\|_{s,2}\|u_0^{k+1}-u_0\|_{s,2}.$$

Note that by the property of mollifiers, $u_0^k, u_0 \in W^{s,2}$ and $\lim_{k\to\infty}\|u_0^k-u_0\|_{s,2}=0$. Then by the Sobolev embedding theorem, u_0^k is bounded so that $A^0(u_0^k)$ is bounded by the continuity of A^0. Thus $B\longrightarrow 0$ as $k\to\infty$. Also

$$\begin{aligned}
A &= c\sup_x\left|\int_0^1\frac{d}{dr}A^0(u_0+r(u_0^k-u_0))dr\right|\|u_0\|_{s,2}^2 \\
&\leqslant c\sup_x\int_0^1|\nabla_u A^0(u_0+r(u_0^k-u_0))|dr|u_0^k-u_0|\|u_0\|_{s,2}^2 \\
&\leqslant c\|u_0^k-u_0\|_{s,2}\|u_0\|_{s,2}^2\longrightarrow 0 \ \text{ as } \ k\to\infty.
\end{aligned}$$

Therefore we have

$$\lim_{k\to\infty}\|u_0^{k+1}\|_{s,A^0(u^k(0))}^2 = \|u_0\|_{s,A^0(0)}^2. \tag{4}$$

And because $u^k \rightharpoonup u$ in $W^{s,2}$,

$$\|u(t)\|_{s,A^0(t)} \leqslant \liminf_{k\to\infty}\|u^{k+1}(t)\|_{s,A^0(t)}. \tag{5}$$

Also (3) is equivalent to

$$\|u^{k+1}(t)\|_{s,A^0(t)}^2 \leqslant \|u_0^{k+1}\|_{s,A^0(u^k(0))}^2 + ct. \tag{6}$$

Thus if

$$\liminf_{k\to\infty}\|u^{k+1}(t)\|_{s,A^0(t)}^2 \leqslant \liminf_{k\to\infty}\|u^{k+1}(t)\|_{s,A^0(u^k(t))}^2 \tag{7}$$

holds then we have

$$\begin{aligned}
\|u(t)\|^2_{s,A^0(t)} &\overset{(5)}{\leqslant} \liminf_{k\to\infty} \|u^{k+1}(t)\|^2_{s,A^0(t)} \\
&\overset{(7)}{\leqslant} \liminf_{k\to\infty} \|u^{k+1}(t)\|^2_{s,A^0(u^k(t))} \\
&\overset{(6)}{\leqslant} \liminf_{k\to\infty} \|u_0^{k+1}\|^2_{s,A^0(u^k(0))} + ct \\
&\overset{(4)}{\leqslant} \|u_0\|^2_{s,A^0(0)} + ct
\end{aligned}$$

which proves the **Lemma 1.** Now we will show that (7) holds for $t \in [0,T]$. Observe that

$$\begin{aligned}
&\left| \sum_{|\alpha|\leqslant s} \int_{\mathbb{R}^n} A^0(u(t))\nabla^\alpha u^{k+1} \cdot \overline{\nabla^\alpha u^{k+1}} - \sum_{|\alpha|\leqslant s} \int_{\mathbb{R}^n} A^0(u^k(t))\nabla^\alpha u^{k+1} \cdot \overline{\nabla^\alpha u^{k+1}} \right| \\
&\leqslant c \sum_{|\alpha|\leqslant s} \int_{\mathbb{R}^n} |A^0(u(t)) - A^0(u^k(t))||\nabla^\alpha u^{k+1}|^2 \\
&\leqslant c \sup_x \left| \int_0^1 \frac{d}{dr} A^0(u^k(t) + r(u(t) - u^k(t)))dr \right| \|u^{k+1}\|^2_{s,2} \\
&\leqslant c \sup_x |\nabla_u A^0(u^k(t) + r(u(t) - u^k(t)))|dr|u(t) - u^k(t)|\|u^{k+1}\|^2_{s,2} \\
&\quad \text{(Note that } u^k \in C^0([0,T], W^{s,2}), u \in C^0([0,T], W^{s-1,2}), u^k(t) \to u(t) \text{ in } W^{s-1,2}) \\
&\leqslant c\|u(t) - u^k(t)\|_{s-1,2} \longrightarrow 0 \ \text{ as } \ k \to \infty
\end{aligned}$$

so that (7) holds. □

Taking limit superior both sides to the **Lemma 1**,

$$\limsup_{t\downarrow 0} \|u(t)\|^2_{s,A^0(t)} \leqslant \|u_0\|^2_{s,A^0(0)}$$

so that $u(t) \longrightarrow u_0$ in $W^{s,2}$ as $t \downarrow 0$. So this proves the regularity of the solution for the system. Now we will improve the existence theorem a little more as follows. □

3. Weakened Dependence with respect to the Time

Theorem 1. Assume the *(assump.)* as before. Let $u_0 \in W^{s,2}$, $s, m \in \mathbb{N}$, $s \geqslant m > 1 + n/2$, $g_1^m := \kappa_m \|u_0\|_{m,2}$, $g_2^m > g_1^m$. Then there exists $T = T(\|u_0\|_{m,2}, g_2^m) > 0$ such that there exists a classical solution $u \in C_b^1([0,T] \times \mathbb{R}^n)$ to the initial value problem with

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |u(t,x)| \leqslant g_2^m,$$

$$u \in C^0([0,T], W^{s,2}) \cap C^1([0,T], W^{s-1,2}).$$

Proof. We will introduce the proof briefly because the proof is almost same as the previous one. By the previous existence theorem, there exists a unique solution $u \in C^0([0,T_m], W^{m,2}) \cap C^1([0,T_m], W^{m-1,2})$, $\sup_{t,x} |u(t,x)| \leqslant g_2^m$, $T_m = T_m(\|u_0\|_{m,2}, g_2^m)$. Define u_0^k, u^k as before. Now our claim is that there exists $R_s, L_s > 0$ and $T \in (0, T_m]$ such that for any $k \in \mathbb{N}_0$, (i)

$|u^k|_{s,T} \leqslant R_s$, (ii) $|\partial_t u^k|_{s-1,T} \leqslant L_s$, and (iii) for any $(t,x) \in [0,T] \times \mathbb{R}^n$, $|u^k(t,x)| \leqslant g_2^m$. Note that for any $k, m \in \mathbb{N}_0$, $u^{k+1} \in C^0([0,\infty), W^{m,2}) \cap C^1([0,\infty), W^{m-1,2}) \cap C^\infty([0,\infty) \times \mathbb{R}^n)$. Our strategy is the induction again. For $k = 0$, it's O.K. with the condition $R_m \geqslant \|u_0\|_{s,2}$. And note that (i) implies (ii) and (ii) implies (iii) by the similar way. Now for the simplicity define $u := u^k$, $v := u^{k+1}$. Then for $t \in [0,T]$, induction hypothesis implies $\|u(t)\|_\infty \leqslant g_2^m$, $\|\nabla u(t)\|_\infty \leqslant \kappa_m \|u(t)\|_{m,2} \leqslant \kappa_m R_m$, and PDE implies $\|\partial_t u(t)\|_\infty \leqslant c(g_2^m) R_m$ so that $|D \cdot A(u(t,x))| \leqslant c(g_2^m) R_m$, $|B(u(t,x))| \leqslant c(g_2^m)$. And define F_α as before. Then by the **Moser type inequality**(as we used before), for $|\alpha| \leqslant s$,

$$\begin{aligned}\|F_\alpha\|_2 \leqslant c \sum_{j=1}^n \left(\|\nabla^1((A^0)^{-1}A^j(u))\|_\infty \|\nabla^{|\alpha|-1}\partial_j v\|_2 + \|\nabla^{|\alpha|}((A^0)^{-1}A^j(u))\|_2 \|\partial_j v\|_\infty \right) \\ + c\left(\|\nabla^1((A^0)^{-1}B(u))\|_\infty \|\nabla^{|\alpha|-1} v\|_2 + \|\nabla^{|\alpha|}((A^0)^{-1}B(u))\|_2 \|v\|_\infty \right).\end{aligned}$$

Note that because $|\alpha| \leqslant s$, $\|\nabla^{|\alpha|-1}\partial_j v\|_2 \leqslant \|v\|_{s,2}$ and $\|\nabla^{|\alpha|-1} v\|_2 \leqslant \|v\|_{s,2}$. And Sobolev embedding implies $\|\partial_j v\|_\infty, \|v\|_\infty \leqslant \|v\|_{m,2}$.
$\|\nabla^1((A^0)^{-1}A^j(u))\|_\infty \leqslant c \sum_{|\alpha|=1} \|\nabla_u((A^0)^{-1}A^j)(u)\|_\infty \|\nabla^\alpha u\|_\infty \leqslant c(g_2^m) R_m$ and $\|\nabla^1((A^0)^{-1}B(u))\|_\infty$ is similar. $\|\nabla^{|\alpha|}((A^0)^{-1}B(u))\|_2 \leqslant c(g_2^m)\|\nabla^{|\alpha|}u\|_2 \leqslant c(g_2^m)\|u\|_{s,2} \leqslant c(g_2^m)R_s$ and $\|\nabla^{|\alpha|}((A^0)^{-1}A^j(u))\|_2$ is similar. Therefore we have

$$\|F_\alpha\|_2 \leqslant c(g_2^m)(R_m\|v\|_{s,2} + R_s\|v\|_{m,2}).$$

Then by the similar way as before,

$$\begin{aligned}\frac{d}{dt}\sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n} A^0(u)\nabla^\alpha v \cdot \overline{\nabla^\alpha v} &\leqslant c(g_2^m)(R_m+1)\|v\|_{s,2}^2 + c\sum_{|\alpha|\leqslant s}\|F_\alpha\|_2\|\nabla^\alpha v\|_2 \\ &\leqslant c(g_2^m)(R_m+1)\|v\|_{s,2}^2 + c(g_2^m)(R_m\|v\|_{s,2} + R_s\|v\|_{m,2})\|v\|_{s,2} \\ &\leqslant c(g_2^m)((R_m+1)\|v\|_{s,2}^2 + R_s R_m\|v\|_{s,2})\end{aligned}$$

because $\|v\|_{m,2} \leqslant R_m$. Integrating both sides from 0 to t and using positive definiteness of A^0,

$$\begin{aligned}\|v(t)\|_{s,2}^2 &\leqslant c(g_2^m)\left(\|v_0\|_{s,2}^2 + \int_0^t ((R_m+1)\|v(r)\|_{s,2}^2 + (R_s\sqrt{R_m})(\sqrt{R_m}\|v(r)\|_{s,2}))dr\right) \\ &\leqslant c(g_2^m)\left(\|v_0\|_{s,2}^2 + \int_0^t (R_m+1)\|v(r)\|_{s,2}^2 dr + R_s^2 R_m t + \int_0^t R_m\|v(r)\|_{s,2}^2 dr\right).\end{aligned}$$

Then the Gronwall's inequality yields

$$\|v(t)\|_{s,2}^2 \leqslant c(g_2^m)(\|v_0\|_{s,2}^2 + TR_s^2R_m)e^{c(g_2^m)(1+2R_m)t}$$

so that

$$\|v(t)\|_{s,2} \leqslant c(g_2^m)(\|v_0\|_{s,2} + \sqrt{TR_m}R_s)e^{c(g_2^m)(1+2R_m)t}$$

and

$$|v|_{s,T} \leqslant c(g_2^m)(\|v_0\|_{s,2} + \sqrt{TR_m}R_s)e^{c(g_2^m)(1+2R_m)T}.$$

Thus choosing $T \in (0, T_m]$ and R_s such that

$$c(g_2^m)\sqrt{TR_m}e^{c(g_2^m)(1+2R_m)T} \leqslant \frac{1}{2}, \quad R_s \geqslant 2c(g_2^m)\|v_0\|_{s,2}e^{c(g_2^m)(1+2R_m)T},\ R_s \geqslant \|u_0\|_{s,2}$$

yields $|v|_{s,T} \leqslant R_s$ which completes the proof. Note that all the other steps are also similar to the previous proof if we change R as R_s and so on. $\square$

References.

[1] R. Racke. *Lectures on Nonlinear Evolution Equations.* Vieweg, 1991.
[2] J. B. Conway. *A Course in Functional Analysis.* Springer, 1990.