M1 Seminar II (Week 5)

High Energy Estimates

D. Kim

2012. 11. 29.

1. Problem

We consider the following initial value problem :

$$y_{tt} - \Delta y = \sum_{i,j=1}^{n} a_{ij}(Dy)\partial_i\partial_j y \quad \text{with} \quad y(t=0) = y_0,\ y_t(t=0) = y_1 \tag{P}$$

where $Dy := (\partial_t y, \partial_1 y, \cdots, \partial_n y)$, $y = y(t, x_1, \cdots, x_n) \in \mathbb{R}$, $a_{ij} = a_{ji} \in C^\infty(\mathbb{R}^{n+1})$, $a_{ij}(0) = 0$ and a_{ij} are real-valued.

2. Preliminaries

Note 1. Because a_{ij} is continuous and $a_{ij}(0) = 0$, using Rayleigh quotients, there exists $m, \eta > 0$ for any $\xi \in \mathbb{C}^n,\ v \in \mathbb{C}^{n+1}$,

$$|v| < \eta \implies |\xi|^2 + \sum_{i,j=1}^{n} a_{ij}(v)\xi_i\overline{\xi_j} \geqslant m|\xi|^2.$$

Note 2. If we define $u := Dy$, then (P) is equivalent to the quasi-linear symmetric hyperbolic system:

$$A^0(u)\partial_t u + \sum_{j=1}^{n} A^j(u)\partial_j u = 0 \quad \text{with} \quad u(t=0) = u_0 := (y_1, \nabla y_0) \tag{P*}$$

for some appropriate A^0 and A^j(as mentioned before).

Note 3. We will use the notation $Du = D^2 y := [\partial_t Dy \quad \partial_1 Dy \quad \cdots \quad \partial_n Dy]^T$, $\|Du\|_p^p := \sum_{i,j=t,1,\cdots,n} \|\partial_i\partial_j y\|_p^p$ and $\|Du\|_\infty := \sum_{i,j=t,1,\cdots,n} \|\partial_i\partial_j y\|_\infty$.

Note 4. In the previous lemma, we have set

$$R := \|u_0\|_{s,2} \max\{1, c_0(g_2)e^{c(g_2)}\}, \quad T_* := \min\left\{\frac{1}{R+1}, \frac{g_2 - g_1}{\kappa_{s-1}R\sqrt{1+R^2}}\right\}$$

respectively. Note that $R \leqslant \|u_0\|_{s,2}(1 + c_0(g_2)e^{c(g_2)}) \leqslant c(g_2)\|u_0\|_{s,2}$. But if $B = O$ ((P*) is the case of $B = O$), T_* can be simpler:

$$T_* = \min\left\{\frac{1}{R}, \frac{g_2 - g_1}{\kappa_{s-1}R\sqrt{1+R^2}}\right\}$$

so that

$$\begin{aligned} T_* &\geqslant c(g_2)\min\left\{\frac{1}{\|u_0\|_{s,2}}, \frac{1}{\|u_0\|_{s,2}\sqrt{1+c(g_2)\|u_0\|_{s,2}^2}}\right\} \\ &\geqslant \frac{c(g_2)}{\|u_0\|_{s,2}\sqrt{1+c(g_2)\|u_0\|_{s,2}^2}} \\ &\geqslant \frac{c(g_2)}{\|u_0\|_{s,2}(1+c(g_2)\|u_0\|_{s,2})} \\ &\geqslant \frac{c(g_2)}{\|u_0\|_{s,2}(1+\|u_0\|_{s,2})}. \end{aligned}$$

Note 5. In the proof of the previous lemma, we have $c(g_2, R) = c(g_2)R \leqslant c(g_2)R(1+R)$. We've set T there such that $0 < T \leqslant T_*$ and $\alpha := \sqrt{c(g_2,R)T}e^{c(g_2,R)T} < 1$. Now let $M = M(g_2)$ satisfying the following equation:

$$Me^{c(g_2)M^2} = \frac{1}{2\sqrt{c(g_2)}}$$

and let $T := \frac{M^2}{R(1+R)}$. Then we have

$$\alpha \leqslant \sqrt{c(g_2)R(1+R)T}e^{c(g_2)R(1+R)T} \leqslant \sqrt{c(g_2)}Me^{c(g_2)M^2} = \frac{1}{2} < 1$$

and

$$T_* \geqslant T \geqslant \frac{c(g_2)}{\|u_0\|_{s,2}(1+\|u_0\|_{s,2})}.$$

Note 6. Note that if $B = O$, in the improved existence theorem, similarly we have

$$T \geqslant \frac{c(g_2^m)}{\|u_0\|_{m,2}(1+\|u_0\|_{m,2})}.$$

3. High Energy Estimates

Now consider the initial value problem (P) and (P*). Let $u_0 \in W^{s,2}$, $s > 2 + n/2$. Because C_0^{s+1} is dense in $W^{s,2}$ and $C_0^{s+1} \subset W^{s+1,2}$, $W^{s+1,2}$ is dense in $W^{s,2}$, that is, there exists $\{u_0^k\} \subset W^{s+1,2}$ such that $\|u_0 - u_0^k\|_{s,2} \longrightarrow 0$ as $k \to \infty$. Then by the local existence theorem, there exists $T_k \geqslant \frac{c(g_2^m)}{\|u_0^k\|_{m,2}(1+\|u_0^k\|_{m,2})}$ such that there exists u^k on $[0, T_k]$ which solves (P*) with the initial condition $u^k(t=0) = u_0^k \in W^{s+1,2}$ for $s+1 \geqslant m > 1 + n/2$. Also there exists $T_0 \geqslant \frac{c(g_2^m)}{\|u_0\|_{m,2}(1+\|u_0\|_{m,2})}$ such that there exists u on $[0, T_0]$ which solves (P*) with the initial condition $u(t=0) = u_0 \in W^{s,2}$ for $s \geqslant m > 1 + n/2$.
Now fix m which satisfies these two arguments both. Because $0 \leqslant \|u_0^k\|_{m,2}, \|u_0\|_{m,2} < \infty$ and $\|u_0^k\|_{m,2} \longrightarrow \|u_0\|_{m,2}$, we have for any $k \in \mathbb{N}$, $T_k \geqslant c > 0$ for some $c > 0$. So now we can define T as

$$T := \inf_{k \in \mathbb{N}_0} T_k > 0$$

to be positive. Note that the first existence theorem is not sufficient because if $u_0 \notin W^{s+1,2}$ then $\|u_0^k\|_{s+1,2} \longrightarrow \infty$ as $k \to \infty$. This means that $\inf_{k\in\mathbb{N}_0} T_k$ may be 0. Now we will introduce an estimate for the initial value problem (P*):

Theorem 1. There exists $c > 0$ which is independent of T and u_0 such that for any $t \in [0, T]$,

$$\|u(t)\|_{s,2} \leqslant c\|u_0\|_{s,2} e^{c\int_0^t \|Du(r)\|_\infty dr}.$$

Proof. Note that by the previous existence theorem, $u^k \in C^0([0,T], W^{s+1,2}) \cap C^1([0,T], W^{s,2})$ and $u \in C^0([0,T], W^{s,2}) \cap C^1([0,T], W^{s-1,2})$. Here $u^k = Dy^k$ which means that $u_1^k = \partial_t y^k$, $u_2^k = \partial_1 y^k, \cdots, u_{n+1}^k = \partial_n y^k$. Also $y_{tt}^k \in C^0([0,T], W^{s,2})$. Now consider the partial differential equation for y^k :

$$y_{tt}^k - \Delta y^k = \sum_{i,j=1}^n a_{ij}(Dy^k)\partial_i\partial_j y^k.$$

Let $0 \leqslant |\alpha| \leqslant s$ and omit the index k. Then using integration by parts,

$$\begin{aligned}
\left\langle \nabla^\alpha \left(\sum_{i,j=1}^n a_{ij}(Dy)\partial_i\partial_j y \right), \nabla^\alpha y_t \right\rangle_{L^2} &= \langle \nabla^\alpha(y_{tt} - \Delta y), \nabla^\alpha y_t \rangle_{L^2} \\
&= \int_{\mathbb{R}^n} (\nabla^\alpha y_{tt})(\nabla^\alpha y_t) - \int_{\mathbb{R}^n} (\nabla^\alpha \Delta y)(\nabla^\alpha y_t) \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} (\partial_t \nabla^\alpha y)^2 + \sum_{j=1}^n \int_{\mathbb{R}^n} (\partial_j\partial_t \nabla^\alpha y)(\partial_j \nabla^\alpha y) \\
&= \frac{1}{2}\frac{d}{dt}\|\nabla^\alpha y_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \sum_{j=1}^n (\nabla^\alpha \partial_j y)^2 \\
&= \frac{1}{2}\frac{d}{dt}\|\nabla^\alpha y_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla^\alpha \nabla y\|_2^2
\end{aligned}$$

holds. Thus we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^\alpha y_t\|_2^2 + \|\nabla^\alpha \nabla y\|_2^2\right) \\
&= \sum_{i,j=1}^n \langle \nabla^\alpha(a_{ij}(Dy)\partial_i\partial_j y), \nabla^\alpha y_t \rangle_{L^2} \\
&= \sum_{i,j=1}^n \langle a_{ij}(Dy)\partial_i\partial_j \nabla^\alpha y, \nabla^\alpha y_t \rangle_{L^2} + \sum_{i,j=1}^n \langle \nabla^\alpha(a_{ij}(Dy)\partial_i\partial_j y) - a_{ij}(Dy)\partial_i\partial_j \nabla^\alpha y, \nabla^\alpha y_t \rangle_{L^2} \\
&=: \text{(I)} + \text{(II)}. \qquad (1)
\end{aligned}$$

Now consider (I). Integrating by parts yields

$$\begin{aligned}
\text{(I)} &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} (a_{ij}(Dy)\nabla^\alpha y_t)(\partial_j\partial_i \nabla^\alpha y) \\
&= -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_j(a_{ij}(Dy)\nabla^\alpha y_t)\partial_i \nabla^\alpha y \\
&= -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_j a_{ij}(Dy)(\partial_i \nabla^\alpha y)(\nabla^\alpha y_t) - \sum_{i,j=1}^n \int_{\mathbb{R}^n} a_{ij}(Dy)(\nabla^\alpha \partial_i y)(\nabla^\alpha \partial_j y_t) \;=:\; \text{(I1)} + \text{(I2)}
\end{aligned}$$

Now (I1) can be estimated by

$$\begin{aligned}
|(\mathrm{I1})| &\leqslant \sum_{i,j=1}^{n}\int_{\mathbb{R}^n}|\partial_j a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\nabla^\alpha y_t)| \\
&\leqslant \sum_{i,j=1}^{n}\int_{\mathbb{R}^n}|(\nabla a_{ij})(Dy)\cdot\partial_j Dy|(|\partial_i\nabla^\alpha y|^2+|\nabla^\alpha y_t|^2) \\
&\leqslant c\sum_{i,j=1}^{n}\|\partial_j Dy\|_\infty(\|\partial_i\nabla^\alpha y\|_2^2+\|\nabla^\alpha\partial_t y\|_2^2) \\
&\leqslant c\|Du^k\|_\infty\|u^k\|_{s,2}^2.
\end{aligned}$$

Now consider (I2). By using the symmetry($a_{ij}=a_{ji}$), (I2) can be written as

$$\begin{aligned}
(\mathrm{I2}) &= -\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}a_{ij}(Dy)(\partial_t\partial_i\nabla^\alpha y)(\partial_j\nabla^\alpha y)-\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\partial_t\partial_j\nabla^\alpha y) \\
&= -\frac{1}{2}\frac{d}{dt}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\partial_j\nabla^\alpha y)+\frac{1}{2}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}\partial_t a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\partial_j\nabla^\alpha y) \\
&=: (\mathrm{I2a})+(\mathrm{I2b})
\end{aligned}$$

Note that the term (I2b) can be estimated similarly with (I1):

$$\begin{aligned}
|(\mathrm{I2b})| &\leqslant c\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}|\partial_t a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\partial_j\nabla^\alpha y)| \\
&\leqslant c\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}|(\nabla a_{ij})(Dy)\cdot\partial_t Dy|(|\partial_i\nabla^\alpha y|^2+|\partial_j\nabla^\alpha y|^2) \\
&\leqslant c\sum_{i,j=1}^{n}\|\partial_t Dy\|_\infty(\|\partial_i\nabla^\alpha y\|_2^2+\|\partial_j\nabla^\alpha y\|_2^2) \\
&\leqslant c\|Du^k\|_\infty\|u^k\|_{s,2}^2.
\end{aligned}$$

Now consider the term (II). If $\alpha=0$ then (II)$=0$. So let $|\alpha|\geqslant 1$. Then by the Schwarz inequality and the Moser-type inequality, we have

$$\begin{aligned}
|(\mathrm{II})| &\leqslant \sum_{i,j=1}^{n}\|\nabla^\alpha(a_{ij}(Dy)\partial_i\partial_j y)-a_{ij}(Dy)\nabla^\alpha\partial_i\partial_j y\|_2\,\|\nabla^\alpha y_t\|_2 \\
&\leqslant c\sum_{i,j=1}^{n}\left(\|\nabla(a_{ij}(Dy))\|_\infty\|\nabla^{|\alpha|-1}\partial_i\partial_j y\|_2+\|\nabla^{|\alpha|}a_{ij}(Dy)\|_2\|\partial_i\partial_j y\|_\infty\right)\|\nabla^\alpha y_t\|_2 \\
&\leqslant c\left(\|\nabla Dy\|_\infty\|\nabla y\|_{s,2}+\|\nabla^{|\alpha|}Dy\|_2\|\nabla^2 y\|_\infty\right)\|\nabla^\alpha y_t\|_2 \\
&\leqslant c\left(\|Du^k\|_\infty\|u^k\|_{s,2}+\|u^k\|_{s,2}\|Du^k\|_\infty\right)\|u^k\|_{s,2} \\
&\leqslant c\|Du^k\|_\infty\|u^k\|_{s,2}^2.
\end{aligned}$$

Therefore the equality (1) becomes (notation : $\cdot_0=\cdot(t=0)$)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|\nabla^\alpha y_t\|_2^2+\|\nabla^\alpha\nabla y\|_2^2\right) &\leqslant c\|Du^k\|_\infty\|u^k\|_{s,2}^2+(\mathrm{I2a}) \\
&= c\|Du^k\|_\infty\|u^k\|_{s,2}^2-\frac{1}{2}\frac{d}{dt}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\partial_j\nabla^\alpha y).
\end{aligned}$$

Now integrating both sides from 0 to t yields

$$\begin{aligned}
&\frac{1}{2}\left(\|\nabla^\alpha y_t\|_2^2+\|\nabla^\alpha\nabla y\|_2^2\right)\\
&\leqslant \frac{1}{2}\left(\|\nabla^\alpha\partial_t y_0\|_2^2+\|\nabla^\alpha\nabla y_0\|_2^2\right)+\int_0^t c\|Du^k\|_\infty\|u^k\|_{s,2}^2\\
&\qquad -\frac{1}{2}\sum_{i,j=1}^n\int_{\mathbb{R}^n}a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\partial_j\nabla^\alpha y)+\frac{1}{2}\sum_{i,j=1}^n\int_{\mathbb{R}^n}a_{ij}(Dy_0)(\partial_i\nabla^\alpha y_0)(\partial_j\nabla^\alpha y_0)\\
&\leqslant c\|u_0^k\|_{s,2}^2+\int_0^t c\|Du^k\|_\infty\|u^k\|_{s,2}^2-\frac{1}{2}\sum_{i,j=1}^n\int_{\mathbb{R}^n}a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\partial_j\nabla^\alpha y)
\end{aligned}$$

or

$$\frac{1}{2}\|\nabla^\alpha y_t\|_2^2\leqslant c\|u_0^k\|_{s,2}^2+\int_0^t c\|Du^k\|_\infty\|u^k\|_{s,2}^2-\frac{1}{2}\sum_{i,j=1}^n\int_{\mathbb{R}^n}a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\partial_j\nabla^\alpha y)-\frac{1}{2}\|\nabla^\alpha\nabla y\|_2^2.$$

Now we can apply the **Note 1** here :

$$|\nabla\nabla^\alpha y|^2+\sum_{i,j=1}^n a_{ij}(Dy)(\partial_i\nabla^\alpha y)(\partial_j\nabla^\alpha y)\geqslant c|\nabla\nabla^\alpha y|^2$$

which yields

$$\frac{1}{2}\|\nabla^\alpha y_t\|_2^2\leqslant c\|u_0^k\|_{s,2}^2+\int_0^t c\|Du^k\|_\infty\|u^k\|_{s,2}^2-c\|\nabla\nabla^\alpha y\|_2^2$$

or

$$\frac{1}{2}\|\nabla^\alpha y_t\|_2^2+c\|\nabla\nabla^\alpha y\|_2^2\leqslant c\|u_0^k\|_{s,2}^2+c\int_0^t\|Du^k\|_\infty\|u^k\|_{s,2}^2.$$

Taking summation for $|\alpha|\leqslant s$ both sides, we have

$$\|u^k(t)\|_{s,2}^2\leqslant c\|u_0^k\|_{s,2}^2+c\int_0^t\|Du^k(r)\|_\infty\|u^k(r)\|_{s,2}^2dr$$

and the Gronwall's inequality implies

$$\|u^k(t)\|_{s,2}\leqslant c\|u_0^k\|_{s,2}e^{c\int_0^t\|Du^k(r)\|_\infty dr}$$

for $0\leqslant t\leqslant T$. Now it remains to investigate the limit $k\to\infty$. Note that PDE implies

$$\begin{aligned}
&A^0(u)\partial_t(u-u^k)+\sum_{j=1}^n A^j(u)\partial_j(u-u^k)\\
&=(A^0(u^k)-A^0(u))\partial_t u^k+\sum_{j=1}^n(A^j(u^k)-A^j(u))\partial_j u^k\ =:\ F_k
\end{aligned}$$

and F_k can be estimated by

$$\begin{aligned}
|F_k| &\leqslant c\left|\int_0^1\frac{d}{dr}A^0(u^k+r(u-u^k))dr\right||\partial_t u^k|+\cdots\\
&\leqslant c\int_0^1\left|\nabla_u A^0(u^k+r(u-u^k))\right|dr|u-u^k||\partial_t u^k|+\cdots\\
&\leqslant c\left(|\partial_t u^k|+\sum_{j=1}^n|\partial_j u^k|\right)|u-u^k|\ \leqslant\ c|u-u^k|
\end{aligned}$$

so that

$$\|F_k\|_2 \leqslant c\|u - u^k\|_2.$$

For the simplicity, let $v := u - u^k$. Then by the symmetry

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} A^0(u)v^k \cdot v^k &= \frac{1}{2}\int_{\mathbb{R}^n} (\partial_t A^0(u))v^k \cdot v^k + \frac{1}{2}\int_{\mathbb{R}^n}\left(\sum_{j=1}^n \partial_j A^j(u)\right)v^k \cdot v^k + \int_{\mathbb{R}^n} F_k \cdot v^k \\ &\leqslant c\|v^k\|_2^2 + c\|F_k\|_2\|v^k\|_2 \\ &\leqslant c\|v^k\|_2^2.\end{aligned}$$

Integrating both sides from 0 to t and using the positive definiteness,

$$\|v^k(t)\|_2^2 \leqslant c\|v_0^k\|_2^2 + c\int_0^t \|v^k(r)\|_2^2 dr$$

and the Gronwall's inequality implies

$$\|v^k(t)\|_2^2 \leqslant c\|v_0^k\|_2^2 e^{ct}$$

which means that

$$\|u^k(t) - u(t)\|_2 \leqslant c\|u_0^k - u_0\|_2 e^{cT}.$$

And because $u_0^k \longrightarrow u_0$ in L^2, we have

$$u^k \longrightarrow u \quad \text{in} \quad C^0([0,T], L^2).$$

And the Gagliardo-Nirenberg's inequality implies

$$u^k \longrightarrow u \quad \text{in} \quad C^0([0,T], W^{s-1,2})$$

also. Because $s - 1 > 1 + n/2$, the Sobolev embedding theorem implies

$$\partial_j u^k \longrightarrow \partial_j u \quad \text{in} \quad C^0([0,T], L^\infty).$$

Also the PDE implies that

$$\partial_t u^k \longrightarrow \partial_t u \quad \text{in} \quad C^0([0,T], W^{s-2,2})$$

and because $s - 2 > n/2$, the Sobolev embedding theorem implies

$$\partial_t u^k \longrightarrow \partial_t u \quad \text{in} \quad C^0([0,T], L^\infty)$$

so that

$$Du^k \longrightarrow Du \quad \text{in} \quad C^0([0,T], L^\infty).$$

Note that the boundedness of $\|u^k(t)\|_{s,2}$ and the Riesz's representation theorem give the weak convergence :

$$u^k(t) \rightharpoonup u(t) \quad \text{in} \quad W^{s,2}$$

for $t \in [0,T]$ and the weak convergence implies that $\|u(t)\|_{s,2} \leqslant \liminf_{k\to\infty} \|u^k(t)\|_{s,2}$. Therefore combining all these results, we have

$$\|u(t)\|_{s,2} \leqslant \liminf_{k\to\infty} \|u^k(t)\|_{s,2} \leqslant c \liminf_{k\to\infty} \|u_0^k\|_{s,2} e^{c\int_0^t \|Du^k(r)\|_\infty dr} = c\|u_0\|_{s,2} e^{c\int_0^t \|Du(r)\|_\infty dr}.$$

□

References.

[1] R. Racke. *Lectures on Nonlinear Evolution Equations.* Vieweg, 1991.
[2] J. B. Conway. *A Course in Functional Analysis.* Springer, 1990.