M1 Seminar II (Week 6)

Weighted a Priori Estimates for Small Data

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2012. 12. 6.

1. Theorem

We consider the following initial value problem :

$$y_{tt} - \Delta y = \sum_{i,j=1}^{n} a_{ij}(Dy)\partial_i\partial_j y \quad \text{with} \quad y(t=0) = y_0,\ y_t(t=0) = y_1 \tag{P}$$

where $Dy := (\partial_t y, \partial_1 y, \cdots, \partial_n y),\ y = y(t, x_1, \cdots, x_n) \in \mathbb{R},\ a_{ij} = a_{ji} \in C^\infty(\mathbb{R}^{n+1}),\ a_{ij}(0) = 0$ and a_{ij} are real-valued. and if we define $u := Dy$, then (P) is equivalent to the quasi-linear symmetric hyperbolic system:

$$A^0(u)\partial_t u + \sum_{j=1}^{n} A^j(u)\partial_j u = 0 \quad \text{with} \quad u(t=0) = u_0 := (y_1, \nabla y_0) \tag{P*}$$

for A^0 and A^j as before. For this problem, we have the following a priori estimates for small data:

Theorem 1. Let $n > 5,\ N_{4/3} > n/2$ and let $s_0, s_1 \in \mathbb{N}$ such that

$$s_1 > \left\lfloor \frac{s_1 + N_{4/3}}{2} \right\rfloor, \quad s_0 \geqslant s_1 + N_{4/3} + 1.$$

Let $u_0 \in W^{s_0,2} \cap W^{s_1+N_{4/3},4/3}$ and let u be the local solution of (P*) on $[0,T]$.
Then there exists $M_0 > 0$ and $\delta_1 > 0$ which are independent of T and u_0 such that if

$$\|u_0\|_{s_0,2} + \|u_0\|_{s_1+N_{4/3},4/3} \leqslant \delta_1$$

then

$$M_{s_1}(T) := \sup_{0\leqslant t\leqslant T} (1+t)^{\frac{n-1}{4}} \|u(t)\|_{s_1,4} \leqslant M_0$$

holds.

Remark 1.
· Note that $s_1 \geqslant N_{4/3} > n/2$ so that $s_1 > n/2 = n/4 + n/4 > 1 + n/4$ ($\because n > 5$).
· Also $s_0 \geqslant s_1 + N_{4/3} + 1 > s_1 + n/4$ and $s_0 \geqslant s_1 + N_{4/3} + 1 > n + 1 \geqslant 2 + n/2$ ($\because n > 5$).
· Because $s_0 > s_1 + n/4$, we have $W^{s_0,2}(\mathbb{R}^n) \hookrightarrow W^{s_1,4}(\mathbb{R}^n)$.
· Because $s_1 > 1 + n/4$, we have $W^{s_1,4}(\mathbb{R}^n) \hookrightarrow C_b^1(\mathbb{R}^n)$.
· $f(Dy, \nabla Dy) = f(u, \nabla u) = \sum_{i,j=1}^{n} a_{ij}(u)\partial_j u_{i+1}$.
· Denote c as various constants which is independent of T and u_0.

2. Lemmas

Lemma 1. $\|f(u,\nabla u)\|_{s_1+N_{4/3},4/3} \leqslant c\|u\|_{s_1,4}\|u\|_{s_0,2}$.

Proof. Note that the right hand side is finite because by the local existence theorem, $u \in W^{s_0,2}$ for the fixed time, and the Sobolev embedding theorem yields $u \in W^{s_1,4}$ also. Note that for $q \geqslant 1$, we have

$$\begin{aligned}\|a_{ij}(u)\|_q = \|a_{ij}(u)-a_{ij}(0)\|_q &= \left\|\int_0^1 \frac{d}{dr}a_{ij}(ru)dr\right\|_q \\ &= \left\|\int_0^1 \nabla_u a_{ij}(ru)\cdot u\right\|_q \leqslant \left\|\int_0^1 |\nabla_u a_{ij}(ru)||u|dr\right\|_q \leqslant c\|u\|_q \qquad \text{(a)}\end{aligned}$$

which will be used for the case of $\alpha = 0$ for the next step. Now by the product rule, we have

$$\begin{aligned}\|f(u,\nabla u)\|_{s_1+N_{4/3},4/3} &= \left\|\sum_{i,j=1}^n a_{ij}(u)\partial_j u_{i+1}\right\|_{s_1+N_{4/3},4/3} \\ &\leqslant \sum_{i,j=1}^n \sum_{0\leqslant|\alpha|\leqslant s_1+N_{4/3}} \|\nabla^\alpha\left(a_{ij}(u)\partial_j u_{i+1}\right)\|_{4/3} \\ &\leqslant c\sum_{i,j=1}^n \sum_{0\leqslant|\alpha|+|\beta|\leqslant s_1+N_{4/3}} \left\|\nabla^\alpha(a_{ij}(u))\nabla^\beta\partial_j u_{i+1}\right\|_{4/3}.\end{aligned}$$

(i) Let $|\alpha| > s_1 > 0$ then $|\beta|+1 \leqslant s_1+N_{4/3}-|\alpha|+1 \leqslant N_{4/3} \leqslant s_1$ $(\because |\alpha| \geqslant s_1+1)$.

$$\begin{aligned}\left\|\nabla^\alpha(a_{ij}(u))\nabla^\beta\partial_j u_{i+1}\right\|_{4/3} &\leqslant \|\nabla^\alpha a_{ij}(u)\|_2\|\nabla^\beta\partial_j u_{i+1}\|_4 \quad \left(\because \frac{1}{4/3}=\frac{1}{2}+\frac{1}{4}\right) \\ &\leqslant \|\nabla_u a_{ij}(u)\|_\infty\|\nabla^\alpha u\|_2\|u\|_{|\beta|+1,4} \\ &\leqslant c\|u\|_{s_0,2}\|u\|_{s_1,4} \quad (\because |\alpha| \leqslant s_1+N_{4/3} \leqslant s_0).\end{aligned}$$

(ii) Let $|\beta|+1 > s_1$ then $|\alpha| \leqslant s_1+N_{4/3}-|\beta| \leqslant N_{4/3} \leqslant s_1$ $(\because |\beta| \geqslant s_1)$.

$$\begin{aligned}\left\|\nabla^\alpha(a_{ij}(u))\nabla^\beta\partial_j u_{i+1}\right\|_{4/3} &\leqslant \|\nabla^\alpha a_{ij}(u)\|_4\|\nabla^\beta\partial_j u_{i+1}\|_2 \\ &\leqslant c\|\nabla^\alpha u\|_4\|u\|_{|\beta|+1,2} \quad (\text{if } \alpha=0 \text{ then use (a)}) \\ &\leqslant c\|u\|_{s_1,4}\|u\|_{s_0,2} \quad (\because |\beta|+1 \leqslant s_1+N_{4/3}+1 \leqslant s_0)\end{aligned}$$

(iii) Let $|\alpha| \leqslant s_1$ and $|\beta|+1 \leqslant s_1$.

$$\begin{aligned}\left\|\nabla^\alpha(a_{ij}(u))\nabla^\beta\partial_j u_{i+1}\right\|_{4/3} &\leqslant \|\nabla^\alpha a_{ij}(u)\|_2\|\nabla^\beta\partial_j u_{i+1}\|_4 \\ &\leqslant c\|\nabla^\alpha u\|_2\|u\|_{|\beta|+1,4} \quad (\text{if } \alpha=0 \text{ then use (a)}) \\ &\leqslant c\|u\|_{s_0,2}\|u\|_{s_1,4}\end{aligned}$$

Thus (i), (ii) and (iii) proves the Lemma 1. □

To state the next lemma, define $w(t)g = (w(t)g)(x)$ be the solution of the initial value problem:

$$v_{tt}-\Delta v = 0 \quad \text{with} \quad v(t=0)=0,\ v_t(t=0)=g.$$

For the simplicity, denote $f(t) = f(u(t,\cdot), \nabla u(t,\cdot))$.

Lemma 2. For $0 \leqslant t \leqslant T$, $u(t)$ can be represented as follows:

$$u(t) = Dw(t)y_1 + D\partial_t w(t)y_0 + \int_0^t Dw(t-r)f(r)dr.$$

Proof. We'll show that

$$\begin{aligned} y(t) &= w(t)y_1 + \partial_t w(t)y_0 + \int_0^t w(t-r)f(r)dr \\ &=: v_1(t) + v_2(t) + v_3(t). \end{aligned}$$

Because $w(t)y_1 = v_1(t)$ and $\partial_t w(t)y_0 = v_2(t)$,

$$\partial_t^2 v_1 - \Delta v_1 = 0,\ v_1(t=0) = 0,\ \partial_t v_1(t=0) = y_1 \tag{1}$$

$$\partial_t^2 v_2 - \Delta v_2 = 0,\ v_2(t=0) = y_0,\ \partial_t v_2(t=0) = \partial_t^2 w(t)y_0(t=0) = \Delta w(t)y_0(t=0) = 0. \tag{2}$$

Then we have

$$\begin{aligned} \partial_t^2 \int_0^t w(t-r)f(r)dr &= \partial_t \left(w(0)f(t) + \int_0^t \partial_t w(t-r)f(r)dr \right) \\ &\quad \left\{ \because h(z_1,z_2) := \int_0^{z_1} w(z_2-r)f(r)dr,\ z_1 = z_1(t), z_2 = z_2(t) \right. \\ &\qquad \partial_t h(z_1,z_2) = \frac{\partial h}{\partial z_1}\frac{dz_1}{dt} + \frac{\partial h}{\partial z_2}\frac{dz_2}{dt} \\ &\qquad\qquad = w(z_2-z_1)f(z_1)\frac{dz_1}{dt} + \int_0^{z_1} \partial_{z_2} w(z_2-r)f(r)dr\frac{dz_2}{dt} \\ &\qquad \partial_t \int_0^t w(t-r)f(r)dr = \partial_t h(t,t) \\ &\qquad\qquad = w(0)f(t) + \int_0^t \partial_t w(t-r)f(r)dr \Big\} \\ &= \partial_t \int_0^t \partial_t w(t-r)f(r)dr \\ &\quad \{\because w(s)f(t) : \text{sol. of } v_{ss} - \Delta v = 0, \\ &\qquad\qquad v(s=0) = 0,\ v_s(s=0) = f(t) \Rightarrow w(0)f(t) = 0\} \\ &= (\partial_t w)(0)f(t) + \int_0^t \partial_t^2 w(t-r)f(r)dr \\ &\quad \left\{ \because \text{ similar by setting } h := \int_0^{z_1} \partial_{z_2} w(z_2-r)f(r)dr \right\} \\ &= f(t) + \int_0^t \partial_t^2 w(t-r)f(r)dr \\ &\quad \{\because\ (\partial_t w)(0)f(t) = (\partial_s w)(s=0)f(t) \Rightarrow (\partial_s w)(0)f(t) = f(t)\} \\ &= f(t) + \int_0^t \Delta w(t-r)f(r)dr \\ &= f(t) + \Delta \int_0^t w(t-r)f(r)dr \end{aligned}$$

which implies that

$$\partial_t^2 v_3 - \Delta v_3 = 0,\ v_3(t=0) = 0,\ \partial_t v_3(t=0) = 0. \tag{3}$$

Thus combining (1), (2), (3), we have

$$\partial_t^2 y - \Delta y = (\partial_t^2 - \Delta)(v_1 + v_2 + v_3) = f \ \text{ with } \ y(t=0) = y_0,\ y_t(t=0) = y_1.$$

Therefore

$$\begin{aligned} u(t) = Dy(t) &= Dw(t)y_1 + D\partial_t w(t) y_0 + D\int_0^t w(t-r) f(r) dr \\ &= Dw(t)y_1 + D\partial_t w(t) y_0 + \int_0^t Dw(t-r) f(r) dr. \end{aligned}$$

The last equality holds because $\partial_j \int_0^t w(t-r)f(r)dr = \int_0^t \partial_j w(t-r)f(r)dr$ and $\partial_t \int_0^t w(t-r)f(r)dr = w(0)f(t) + \int_0^t \partial_t w(t-r)f(r)dr = \int_0^t \partial_t w(t-r)f(r)dr$. □

Remark 2. For $d > 1$,

$$\sup_{0 \leqslant t < \infty} \int_0^t (1+t)^d (1+t-r)^{-d} (1+r)^{-d} dr \leqslant c = c(d) < \infty.$$

Because the integrand is symmetric with respect to $r = t/2$,

$$\begin{aligned} \int_0^t (1+t)^d (1+t-r)^{-d} (1+r)^{-d} dr &= 2\int_0^{t/2} (1+t)^d (1+t-r)^{-d} (1+r)^{-d} dr \\ &\leqslant 2\int_0^{t/2} (1+t)^d (1+t-t/2)^{-d} (1+r)^{-d} dr \\ &\leqslant 2^{d+1} \int_0^{t/2} (1+r)^{-d} dr \\ &= \frac{2^{d+1}}{-d+1} \left((1+t/2)^{-d+1} - 1 \right) \ \leqslant \ \frac{2^{d+1}}{d-1}. \end{aligned}$$

□

Lemma 3. (*$L^p - L^q$ Decay Estimates for the Linear Wave Equations*)

Let $2 \leqslant q \leqslant \infty,\ 1/p + 1/q = 1,\ N_p > n(1-2/q)$. Then there exists $c = c(q,n) > 0$ such that for any $g \in W^{N_p,p}$ and for all $t \geqslant 0$,

$$\|Dw(t)g\|_q \leqslant c(1+t)^{-\frac{n-1}{2}(1-\frac{2}{q})} \|g\|_{N_p,p}$$

holds. We are now considering the case of $p = 4/3$ so that $q = 4$. Thus there exists $c > 0$ such that for any $g \in W^{N_{4/3},4/3}$ and for all $t \geqslant 0$,

$$\|Dw(t)g\|_4 \leqslant c(1+t)^{-\frac{n-1}{4}} \|g\|_{N_{4/3},4/3}.$$

We have proved this decay estimates in the last seminars, so the proof is omitted here. And also the following estimate plays an important role in the proof of the theorem. □

Lemma 4. (*High Energy Estimates*)

Let $s > 2 + n/2$. There is a constant $c > 0$ which is independent of T and u_0 such that for all $t \in [0, T]$, the local solution u satisfies

$$\|u(t)\|_{s,2} \leqslant c\|u_0\|_{s,2} e^{c\int_0^t \|Du(r)\|_\infty dr}.$$

We have shown this estimate in the last week. □

3. Proof of the Theorem

Note that by the Lemma 2, u can be written by

$$u(t) = Dw(t)y_1 + Dw(t)y_0 + \int_0^t Dw(t-r)f(r)dr \;=:\; u^1(t) + u^2(t) + u^3(t).$$

And for the simplicity let

$$d := \frac{n-1}{4} > 1.$$

Also remember that our assumption of the theorem, $\|u_0\|_{s_0,2} + \|u_0\|_{s_1+N_{4/3},4/3} < \delta_1$ where $u_0 = (y_1, \nabla y_0)$.

Proof of the Thm. From now on, we will estimate $\|u^k(t)\|_{s_1,4}$ for $k = 1, 2, 3$ respectively:

$$\begin{aligned}
\|u^1(t)\|_{s_1,4} &= \|Dw(t)y_1\|_{s_1,4} \leqslant \sum_{|\alpha|\leqslant s_1} \|\nabla^\alpha Dw(t)y_1\|_4 = \sum_{|\alpha|\leqslant s_1} \|Dw(t)\nabla^\alpha y_1\|_4 \\
&\leqslant \sum_{|\alpha|\leqslant s_1} c(1+t)^{-d}\|\nabla^\alpha y_1\|_{N_{4/3},4/3} \quad (\because u_0 \in W^{s_1+N_{4/3},4/3} \Rightarrow \nabla^\alpha y_1 \in W^{N_{4/3},4/3}) \\
&\leqslant c(1+t)^{-d}\|y_1\|_{s_1+N_{4/3},4/3} \\
&\leqslant c(1+t)^{-d}\delta_1
\end{aligned}$$

$$\begin{aligned}
\|u^2(t)\|_{s_1,4} &= \|D\partial_t w(t)y_0\|_{s_1,4} = \|(\partial_t^2 w(t)y_0, \nabla\partial_t w(t)y_0)\|_{s_1,4} \\
&= \|(\Delta w(t)y_0, \nabla\partial_t w(t)y_0)\|_{s_1,4} = \left\| \left(\sum_{j=1}^n \partial_j^2 w(t)y_0, \partial_t \nabla w(t)y_0 \right) \right\|_{s_1,4} \\
&= \left\| \left(\sum_{j=1}^n \partial_j w(t)\partial_j y_0, \partial_t(w(t)\nabla y_0) \right) \right\|_{s_1,4} \leqslant c\sum_{j=1}^n \|Dw(t)\partial_j y_0\|_{s_1,4} \\
&\leqslant c\sum_{|\alpha|\leqslant s_1}\sum_{j=1}^n \|\nabla^\alpha Dw(t)\partial_j y_0\|_4 = c\sum_{|\alpha|\leqslant s_1}\sum_{j=1}^n \|Dw(t)\nabla^\alpha \partial_j y_0\|_4 \\
&\leqslant c\sum_{|\alpha|\leqslant s_1}\sum_{j=1}^n (1+t)^{-d}\|\nabla^\alpha \partial_j y_0\|_{N_{4/3},4/3} \quad (\because \nabla y_0 \in W^{s_1+N_{4/3},4/3}) \\
&\leqslant c(1+t)^{-d}\|\nabla y_0\|_{s_1+N_{4/3},4/3} \\
&\leqslant c(1+t)^{-d}\delta_1
\end{aligned}$$

$$
\begin{aligned}
\|u^3(t)\|_{s_1,4} &= \left\| \int_0^t Dw(t-r)f(r)dr \right\|_{s_1,4} \leqslant \int_0^t \|Dw(t-r)f(r)\|_{s_1,4}dr \\
&\leqslant \int_0^t \sum_{|\alpha|\leqslant s_1} \|Dw(t-r)\nabla^\alpha f(r)\|_4 dr \\
&\leqslant c\int_0^t \sum_{|\alpha|\leqslant s_1} (1+t-r)^{-d}\|\nabla^\alpha f(r)\|_{N_{4/3},4/3}dr \qquad (\because \text{ Lemma 1}) \\
&\leqslant c\int_0^t (1+t-r)^{-d}\|f(r)\|_{s_1+N_{4/3},4/3}dr \\
&\leqslant c\int_0^t (1+t-r)^{-d}\|u(r)\|_{s_1,4}\|u(r)\|_{s_0,2}dr \qquad (\because \text{ Lemma 1}) \\
&\leqslant c\int_0^t (1+t-r)^{-d}\|u(r)\|_{s_1,4}\|u_0\|_{s_0,2}e^{c\int_0^t \|Du(\tau)\|_\infty d\tau}dr \qquad (\because \text{ Lemma 4}) \\
&\quad (\|u_0\|_{s_0,2} \leqslant \delta_1 \text{ and } 0 \leqslant r \leqslant t : M_{s_1}(t) \geqslant (1+r)^d\|u(r)\|_{s_1,4}) \\
&\leqslant c\delta_1 e^{c\int_0^t \|Du(\tau)\|_\infty d\tau}\int_0^t (1+t-r)^{-d}(1+r)^{-d}M_{s_1}(t)dr \\
&= c\delta_1 e^{c\int_0^t \|Du(\tau)\|_\infty d\tau}M_{s_1}(t)(1+t)^{-d}\int_0^t (1+t)^d(1+t-r)^{-d}(1+r)^{-d}dr \\
&\leqslant c\delta_1(1+t)^{-d}M_{s_1}(t)e^{c\int_0^t \|Du(\tau)\|_\infty d\tau}. \qquad (\because \text{ Remark 2})
\end{aligned}
$$

And because $s_1 > 1+n/4$, we have $\|\nabla u\|_\infty \leqslant c\|u\|_{s_1,4}$. Also the partial differential equation implies that $\|\partial_t u\|_\infty \leqslant c\|\nabla u\|_\infty \leqslant c\|u\|_{s_1,4}$ so that $\|Du\|_\infty \leqslant c\|u\|_{s_1,4}$. Thus we have

$$
\begin{aligned}
\int_0^t \|Du(\tau)\|_\infty d\tau &\leqslant \int_0^t c\|u(\tau)\|_{s_1,4}d\tau \\
&= c\int_0^t (1+\tau)^{-d}(1+\tau)^d\|u(\tau)\|_{s_1,4}d\tau \\
&\leqslant c\int_0^t (1+\tau)^{-d}M_{s_1}(t)d\tau \\
&\leqslant cM_{s_1}(t).
\end{aligned}
$$

Therefore

$$
\|u^3(t)\|_{s_1,4} \leqslant c\delta_1(1+t)^{-d}M_{s_1}(t)e^{cM_{s_1}(t)}
$$

holds and we finally obtain the estimate for u which is

$$
\|u(t)\|_{s_1,4} \leqslant \sum_{j=1}^3 \|u^j(t)\|_{s_1,4} \leqslant c\delta_1(1+t)^{-d} + c\delta_1(1+t)^{-d}M_{s_1}(t)e^{cM_{s_1}(t)}
$$

or using the definition of $M_{s_1}(t)$,

$$
M_{s_1}(t) \leqslant (1+t)^d\|u(t)\|_{s_1,4} \leqslant c\delta_1\left(1 + M_{s_1}(t)e^{cM_{s_1}(t)}\right) \tag{M}
$$

for all $t \in [0,T]$.

Without loss of generality, we may assume $c > \tilde{\kappa}$ where $\tilde{\kappa}$ denotes the embedding constant in $W^{s_0,2} \hookrightarrow W^{s_1,4}$. Define $\phi : [0,\infty) \to \mathbb{R}^n$ by $\phi(x) := c\delta_1(1 + xe^{cx}) - x$ shown in the Figure 1 below.

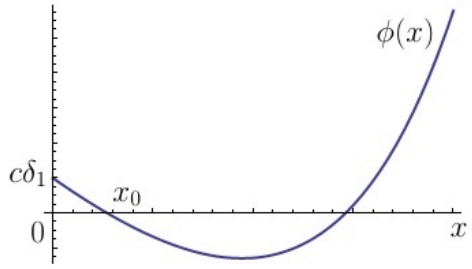

Figure 1: $\phi(x)$ for sufficiently small δ_1

Let x_0 be the first zero of ϕ so that

$$0 = \phi(x_0) = c\delta_1(1 + x_0e^{cx_0}) - x_0$$

which implies

$$\delta_1 = \frac{x_0}{c(1 + x_0e^{cx_0})} < \frac{x_0}{c}.$$

Note that

$$M_{s_1}(0) = \|u_0\|_{s_1,4} \leqslant \tilde{\kappa}\|u_0\|_{s_0,2} \leqslant \tilde{\kappa}\delta_1 < x_0$$

and the inequality (M) implies

$$\phi(M_{s_1}(t)) \geqslant 0$$

for all $t \in [0,T]$. Thus by the continuity assertion, we have

$$M_{s_1}(t) \leqslant x_0 =: M_0$$

for all $t \in [0,T]$ which proves the theorem. □

References.

[1] R. Racke. *Lectures on Nonlinear Evolution Equations.* Vieweg, 1991.
[2] S. Klainerman, G. Ponce. Global, Small Amplitude Solutions to Nonlinear Evolution Equations. *Comm. Pure Appl. Math.* 36, 1983.