M1 Seminar II (Week 7)

A Global Existence Theorem for Nonlinear Wave Equations

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2012. 12. 20.

1. Theorem

We consider the following initial value problem

$$y_{tt} - \Delta y = \sum_{i,j=1}^{n} a_{ij}(Dy)\partial_i\partial_j y \quad \text{with} \quad y(t=0) = y_0,\ y_t(t=0) = y_1 \tag{P}$$

with $f \in C^\infty(\mathbb{R}^{(n+1)^2}, \mathbb{R})$, $\exists \alpha \in \mathbb{N}$ such that $f(u, \nabla u) = O((|u| + |\nabla u|)^{\alpha+1})$ as $|u| + |\nabla u| \to 0$ where $u := Dy$ is the corresponding solution of the quasi-linear symmetric hyperbolic system with $u_0 = (y_1, \nabla y_0)$. With these assumptions the following theorem holds.

Theorem 1. Let $\frac{1}{\alpha}\left(1 + \frac{1}{\alpha}\right) < \frac{n-1}{2}$. Then there exists $s_0 > 1 + n/2$ and $\delta > 0$ such that the following holds:
If $u_0 \in W^{s,2} \cap W^{s,p}$ with $s \geqslant s_0$, $p = \frac{2\alpha+2}{2\alpha+1}$, $\|u_0\|_{s,2} + \|u_0\|_{s,p} < \delta$ then there exists a unique solution y of the problem (P) with

$$u \in C^0([0,\infty), W^{s,2}) \cap C^1([0,\infty), W^{s-1,2}).$$

Moreover

$$\|u(t)\|_\infty + \|u(t)\|_{2\alpha+2} = O\left(t^{-\frac{n-1}{2}\frac{\alpha}{\alpha+1}}\right), \qquad \|u(t)\|_{s,2} = O(1)$$

as $t \to \infty$.

2. Lemmas

Lemma 1. (*High Energy Estimates*)
Let $\alpha = 1,\ s > 2 + n/2$ then there exists $c > 0$ which is independent of T, u_0 such that for any $t \in [0, T]$,

$$\|u(t)\|_{s,2} \leqslant c\|u_0\|_{s,2} e^{\int_0^t \|Du(\tau)\|_\infty d\tau}.$$

Lemma 2. (*A Priori Estimates*)
Let $\alpha = 1,\ n > 5,\ s_0, s_1 \in \mathbb{N}$ such that $s_1 > \left[\frac{s_1 + N_{4/3}}{2}\right],\ s_0 \geqslant s_1 + N_{4/3} + 1$. Let $u_0 \in W^{s_0,2} \cap W^{s_1+N_{4/3},4/3}$. Then there exists $M_0 > 0$ and $\delta_1 > 0$ which is independent of u_0 and T such that if

$$\|u_0\|_{s_0,2} + \|u_0\|_{s_1+N_{4/3},4/3} \leqslant \delta_1$$

then

$$M_{s_1}(T) := \sup_{0\leqslant t\leqslant T} (1+t)^{\frac{n-1}{4}} \|u(t)\|_{s_1,4} \leqslant M_0.$$

Note that these two lemmas are for the special case of the **Theorem 1**. Especially for the **Lemma 2**, we chose $p = 4/3$. We will generalize these two lemmas for arbitrary α and will complete the proof of the **Theorem 1** in the end of this paper. To introduce the concept of the proof, we first consider this special case.

3. Proof of the Theorem

Let $\alpha = 1$ and assume $f(Dy, \nabla Dy) = \sum_{i,j=1}^n a_{ij}(Dy)\partial_i\partial_j y$ (or equivalently $f(u, \nabla u) = \sum_{i,j=1}^n a_{ij}(u)\partial_j u_{i+1}$). Then we can confirm that

$$\begin{aligned}
|f(u,\nabla u)| &\leqslant \sum_{i,j=1}^n |a_{ij}(u)||\partial_j u_{i+1}| \leqslant \sum_{i,j=1}^n |a_{ij}(u) - a_{ij}(0)||\nabla u| \\
&= \sum_{i,j=1}^n \left| \int_0^1 \partial_r(a_{ij}(ru))dr \right| |\nabla u| \leqslant \sum_{i,j=1}^n \int_0^1 |\nabla_u a_{ij}(ru)|\, dr|u||\nabla u| \\
&\leqslant c(g_2)|u||\nabla u|
\end{aligned}$$

for $|u| \leqslant g_2$ so that $f(u, \nabla u) = O((|u| + |\nabla u|)^2)$ as $|u| + |\nabla u| \to 0$.
Now let $s \geqslant s_0$. Then by the help of the **Lemma 1** and **Lemma 2**, we have

$$\begin{aligned}
\|u(t)\|_{s,2} &\leqslant c\|u_0\|_{s,2} e^{c\int_0^t \|Du(\tau)\|_\infty d\tau} \\
&\leqslant c\|u_0\|_{s,2} e^{cM_{s_1}(t)} \\
&\leqslant c\|u_0\|_{s,2} e^{cM_0} \\
&\leqslant K\|u_0\|_{s,2}
\end{aligned}$$

for $0 \leqslant t \leqslant T$ and $K := ce^{cM_0}$. Now chooses δ such that $0 < \delta < \delta_1/K$ then

$$\|u(T)\|_{s,2} \leqslant K\|u_0\|_{s,2} < K\delta < \delta_1.$$

Applying the local existence theorem for the quasi-linear symmetric hyperbolic system at the initial time T, then we can extend the solution u onto $[T, T + T_1(\delta_1)]$. Then $\|u(t)\|_{s,2} \leqslant K\|u_0\|_{s,2}$ for $0 \leqslant t \leqslant T + T_1(\delta_1)$ so that $\|u(T + T_1(\delta_1))\|_{s,2} \leqslant K\|u_0\|_{s,2} < K\delta < \delta_1$. Thus we can extend the solution u onto $[T + T_1(\delta_1), T + 2T_1(\delta_1)]$ with the initial time $T + T_1(\delta_1)$. By repeating this process, we have a global solution u with

$$u \in C^0([0,\infty), W^{s,2}) \cap C^1([0,\infty), W^{s-1,2}).$$

Note that this regularities follows by the local existence theorem. Also

$$\|u(t)\|_{s,2} \leqslant K\|u_0\|_{s,2} < K\delta < \delta_1$$

for $0 \leqslant t < \infty$ and

$$\begin{aligned}
\|u(t)\|_\infty + \|u(t)\|_4 &\leqslant c\|u(t)\|_{s_1,4} \\
&\leqslant cM_{s_1}(t)(1+t)^{-\frac{n-1}{4}} \\
&\leqslant cM_0(1+t)^{-\frac{n-1}{4}}
\end{aligned}$$

by the help of the Sobolev embedding theorem and the **Lemma 2**. This proves the **Theorem 1** for the special case with $\alpha = 1$.

Now let $\alpha \geqslant 1$ be arbitrary given and let $f(u, \nabla u) = \sum_{i,j=1}^{n} a_{ij}(u)\partial_j u_{i+1}$ for $a_{ij} \in C^\infty$. Assume

$$a_{ij}(u) = O(|u|^\alpha) \text{ as } |u| \to 0 \tag{a}$$

so that $f(u, \nabla u) = O((|u| + |\nabla u|)^{\alpha+1})$ as $|u| + |\nabla u| \to 0$. Because $a_{ij} \in C^\infty(\mathbb{R}^{n+1})$, the Taylor series gives

$$a_{ij}(u) = \sum_{|\zeta|=0}^{\infty} z_{ij,\zeta}(u)u^\zeta$$

with $|\nabla_u^k z_{ij,\zeta}(u)| = O(1)$ for $k = 0, 1, 2, \cdots$ where the multi-index $\zeta = (\zeta_1, \cdots, \zeta_{n+1})$. Note that because we now consider a small u, with the assumption (a), we have the following simple expression for a_{ij}:

$$a_{ij}(u) = \sum_{|\zeta|=\alpha}^{\infty} z_{ij,\zeta}(u)u^\zeta \leqslant c \sum_{|\zeta|=\alpha} z_{ij,\zeta}(u)u^\zeta =: c \sum_{|\zeta|=\alpha} z(u)u^\zeta.$$

Therefore we have the following remarks.

Remark 1. $\nabla_u a_{ij}(u) = O(|u|^{\alpha-1})$ as $|u| \to 0$ because

$$\begin{aligned} |\nabla_u a_{ij}(u)| &\leqslant c \sum_{|\zeta|=\alpha} |\nabla_u (z(u)u^\zeta)| = c \sum_{|\zeta|=\alpha} \left|(\nabla_u z(u))u^\zeta + z(u)\nabla_u u^\zeta\right| \\ &\leqslant c \sum_{|\zeta|=\alpha} |u^\zeta| + c \sum_{|\zeta|=\alpha-1} |u^\zeta| \leqslant c \sum_{|\zeta|=\alpha-1} |u^\zeta|. \end{aligned}$$

Remark 2. For $\beta = (\beta_1, \cdots, \beta_n)$,

$$\begin{aligned} |\nabla_x^\beta (a_{ij}(u))| &\leqslant \sum_{|\zeta|=\alpha} |\nabla_x^\beta (z(u)u^\zeta)| \\ &\leqslant c \sum_{|\zeta|=\alpha} \sum_{|\beta_1|+|\beta_2|=|\beta|} \left|(\nabla_x^{\beta_1} z(u))u^\zeta + z(u)\nabla_x^{\beta_2} u^\zeta\right| \\ &\leqslant c \sum_{|\zeta|=\alpha} |u^\zeta| + c \sum_{k=0}^{|\beta|} |\nabla^k u|^\alpha. \end{aligned}$$

Then the proof of the **Lemma 1** becomes:

$$\begin{aligned} |\text{I.1}| &\leqslant \sum_{i,j=1}^{n} \|(\nabla_u a_{ij})(u) \cdot \partial_i u\|_\infty (\|\nabla^\beta \partial_j y\|_2^2 + \|\nabla^\beta y_t\|_2^2) \\ &\leqslant c \sum_{i,j=1}^{n} \|u\|_\infty^{\alpha-1} \|\partial_i u\|_\infty \|u\|_{s,2}^2 \\ &\leqslant c \|\overline{D} u\|_\infty^\alpha \|u\|_{s,2}^2 \end{aligned}$$

where $\overline{D} := (1, \partial_t, \partial_1, \cdots, \partial_n)$. And |I.2.b| and |II| are similar to |I.1|. And the other steps are exactly same as before. Thus we conclude that

$$\|u(t)\|_{s,2} \leqslant c\|u_0\|_{s,2} e^{\int_0^t \|\overline{D}u(\tau)\|_\infty^\alpha d\tau} \qquad \textbf{(Lemma 1*)}$$

for the arbitrary α.
Now let s_1 be sufficiently large and let

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \frac{\alpha}{q} + \frac{1}{2} = \frac{1}{p}$$

or

$$p = \frac{2\alpha + 2}{2\alpha + 1}, \quad q = 2(\alpha + 1).$$

Let $N_p > n(1 - 2/q)$ and $\frac{n-1}{2} > \frac{1}{\alpha}(1 + \frac{1}{\alpha})$. Also let $s_1 > [\frac{s_1+N_p}{2}]$, $s_0 \geqslant s_1 + N_p + 1$, $u_0 \in W^{s_0,2} \cap W^{s_1+N_p,p}$. Then the proof of the **Lemma 2** becomes:

$$\|f(u, \nabla u)\|_{s_1+N_p,p} \leqslant c \sum_{i,j=1}^{n} \sum_{|\beta|+|\gamma| \leqslant s_1+N_p} \|(\nabla^\beta a_{ij}(u))\nabla^\gamma \partial_j u_{i+1}\|_p$$

(i) $|\gamma| + 1 > s_1$ $(\Rightarrow |\beta| \leqslant s_1)$

$$\begin{aligned}
\|(\nabla^\beta a_{ij}(u))\nabla^\gamma \partial_j u_{i+1}\|_p &\leqslant c \left\| \left(\sum_{|\zeta|=\alpha} |u^\zeta| + \sum_{k=0}^{|\beta|} |\nabla^k u|^\alpha \right) \nabla^\gamma \partial_j u_{i+1} \right\|_p \quad (\because \textbf{Remark}) \\
&\leqslant c \left(\sum_{|\zeta|=\alpha} \|u^\zeta\|_{\frac{q}{\alpha}} + c \sum_{k=0}^{|\beta|} \||\nabla^k u|^\alpha\|_{\frac{q}{\alpha}} \right) \|\nabla^{|\gamma|+1} u\|_2 \quad (\because \frac{\alpha}{q} + \frac{1}{2} = \frac{1}{p}) \\
&\leqslant c \left(\||u|^\alpha\|_{q/\alpha} + \sum_{|\beta| \leqslant s_1} \|(\nabla^\beta u)^\alpha\|_{q/\alpha} \right) \|u\|_{s_0,2} \quad (\because |u^\zeta| \leqslant c|u|^\alpha) \\
&\leqslant c \left(\|u\|_q^\alpha + \|u\|_{s_1,q}^\alpha \right) \|u\|_{s_0,2} \\
&\leqslant c\|u\|_{s_1,q}^\alpha \|u\|_{s_0,2}
\end{aligned}$$

The other cases (ii) $|\beta| \leqslant s_1$, $|\gamma| + 1 \leqslant s_1$ and (iii) $|\beta| > s_1 (\Rightarrow |\gamma| + 1 \leqslant s_1)$ are similar to the case (i). Therefore we have

$$\|f(u, \nabla u)\|_{s_1+N_p,p} \leqslant c\|u\|_{s_1,q}^\alpha \|u\|_{s_0,2}. \qquad \textbf{(Lemma 2*)}$$

Now let

$$M_{s_1}(T) := \sup_{0 \leqslant t \leqslant T} (1 + t)^{\frac{n-1}{2}(1-\frac{2}{q})} \|u(t)\|_{s_1,q},$$

$$d := \alpha \frac{n-1}{2} \left(1 - \frac{2}{q} \right) = \alpha \frac{\alpha}{\alpha + 1} \frac{n-1}{2}$$

so that $d > 1$ by the assumption $\frac{1}{\alpha}\frac{\alpha+1}{\alpha} < \frac{n-1}{2}$. Similarly as before, if $\|u_0\|_{s_0,2} + \|u_0\|_{s_1+N_p,p} \leqslant \delta_1$ then the $L^p - L^q$ estimate for the linear wave equations yields

$$\|u^1(t)\|_{s_1,q} \leqslant c(1 + t)^{-\frac{n-1}{2}(1-\frac{2}{q})} \|y_1\|_{s_1+N_p,p} \leqslant c(1 + t)^{-d/\alpha} \delta_1.$$

Similarly $\|u^2(t)\|_{s_1,q} \leqslant c(1+t)^{-d/\alpha}\delta_1$. And u^3 can be estimated by

$$\begin{aligned}
\|u^3(t)\|_{s_1,q} &\leqslant c\int_0^t (1+t-r)^{d/\alpha}\|f(r)\|_{s_1+N_p,p}dr \ \ (\because L^p-L^q \text{ Estimates}) \\
&\leqslant c\int_0^t (1+t-r)^{-d/\alpha}\|u(r)\|_{s_1,q}^{\alpha}\|u(r)\|_{s_0,2}dr \ \ (\because \textbf{Lemma 2*}) \\
&\leqslant c\int_0^t (1+t-r)^{-d/\alpha}\|u(r)\|_{s_1,q}^{\alpha}\|u_0\|_{s_0,2}e^{c\int_0^t \|\overline{D}u(\tau)\|_\infty^\alpha d\tau}dr \ \ (\because \textbf{Lemma 1*}) \\
&\leqslant c\delta_1 e^{c\int_0^t \|\overline{D}u(\tau)\|_\infty^\alpha d\tau}M_{s_1}^{\alpha}(t)(1+t)^{-d/\alpha}\int_0^t (1+t)^{d/\alpha}(1+t-r)^{-d/\alpha}(1+r)^{-d}dr \\
&\leqslant c\delta_1(1+t)^{-d/\alpha}M_{s_1}^{\alpha}(t)e^{c\int_0^t \|\overline{D}u(\tau)\|_\infty^\alpha d\tau} \\
&\leqslant c\delta_1(1+t)^{-d/\alpha}M_{s_1}^{\alpha}(t)e^{c\int_0^t \|u(\tau)\|_{s_1,q}^\alpha d\tau} \ \ (\because \text{for sufficiently large } s_1 > 1+n/q) \\
&\leqslant c\delta_1(1+t)^{-d/\alpha}M_{s_1}^{\alpha}(t)e^{c\int_0^t (1+\tau)^{-d}(1+\tau)^d\|u(\tau)\|_{s_1,q}^\alpha d\tau} \\
&\leqslant c\delta_1(1+t)^{-d/\alpha}M_{s_1}^{\alpha}(t)e^{c\int_0^t (1+\tau)^{-d}M_{s_1}^{\alpha}(t)d\tau} \\
&\leqslant c\delta_1(1+t)^{-d/\alpha}M_{s_1}^{\alpha}(t)e^{cM_{s_1}^{\alpha}(t)}.
\end{aligned}$$

Therefore

$$\|u(t)\|_{s_1,q} \leqslant c(1+t)^{-d/\alpha}\delta_1(1+M_{s_1}^{\alpha}(t)e^{cM_{s_1}^{\alpha}(t)})$$

or

$$M_{s_1}^{\alpha}(t) \leqslant (1+t)^d\|u(t)\|_{s_1,q}^{\alpha} \leqslant c\delta_1^{\alpha}(1+(M_{s_1}^{\alpha}(t))^{\alpha}e^{cM_{s_1}^{\alpha}(t)})$$

for $0 \leqslant t \leqslant T$. Thus if we define $\phi(x) := c\delta_1^{\alpha}(1+x^{\alpha}e^{cx}) - x$ for sufficiently small δ_1, similarly we can easily show that $M_{s_1}^{\alpha}(t) \leqslant c(\delta_1)$ for $0 \leqslant t \leqslant T$. Thus there exists $M_0 > 0$ and $\delta_1 > 0$ such that if $\|u_0\|_{s_0,2} + \|u_0\|_{s_1+N_p,p} \leqslant \delta_1$ then $M_{s_1}(T) \leqslant M_0$.

Now let $s \geqslant s_0$. Then

$$\|u(t)\|_{s,2} \leqslant c\|u_0\|_{s,2}e^{c\int_0^t \|\overline{D}u(\tau)\|_\infty^\alpha d\tau} \leqslant c\|u_0\|_{s,2}e^{cM_{s_1}^{\alpha}(t)} \leqslant c\|u_0\|_{s,2}e^{cM_0^{\alpha}}$$

holds. Thus by setting $K := e^{cM_0^{\alpha}}$ the proof of the existence of the global solution is exactly same as before (choose $\delta < \delta_1/K$). Moreover we have for $0 \leqslant t < \infty$, $\|u(t)\|_{s,2} \leqslant K\delta < \delta_1$ and the Sobolev embedding theorem implies

$$\begin{aligned}
\|u(t)\|_\infty + \|u(t)\|_{2\alpha+2(=q)} &\leqslant c\|u(t)\|_{s_1,q} \\
&\leqslant cM_{s_1}(t)(1+t)^{-\frac{n-1}{2}(1-\frac{2}{q})} \\
&\leqslant cM_0(1+t)^{-\frac{n-1}{2}\frac{\alpha}{\alpha+1}}
\end{aligned}$$

which proves the **Theorem 1**. □

References

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