Seminar Handout (Week 10)

Some Inequalities

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1. Notatins and Lemmas

Notation. For $i \in \mathbb{N}_0,\ 1 \leqslant q < \infty$,

$$\|\nabla^i v\|_q := \left(\sum_{|\alpha|=i} \|\nabla^\alpha v\|_q^q \right)^{1/q}$$

Lemma 4.6. Let $1 \leqslant p, r \leqslant \infty,\ f \in C^2(\mathbb{R}) \cap L^r(\mathbb{R}),\ f'' \in L^p(\mathbb{R}),\ 2/q = 1/p + 1/r$. Then we have $f' \in L^q(\mathbb{R})$ and there is a constant $c = c(q) > 0$ such that

$$||f'||_q \leqslant c||f''||_p^{1/2}||f||_r^{1/2}.$$

2. Theorems

Theorem 4.4. Let $1 \leqslant r, p \leqslant \infty,\ m \in \mathbb{N}$. Then there is a constant $c > 0$ such that for all $w \in W^{m,p} \cap L^r$ the inequality

$$||\nabla^j w||_q \leqslant c||\nabla^m w||_p^{j/m}||w||_r^{1-j/m}$$

holds, where $j \in \{0, 1, \cdots, m\}$ and

$$\frac{1}{q} = \frac{j}{mp} + (1 - \frac{j}{m})\frac{1}{r}.$$

Proof. First, we will prove this for $v = J_\epsilon w$ by induction on m.
(1) $m = 2$: (i) $r, p \neq \infty$, (ii) $r = \infty, p \neq \infty$ or $r \neq \infty, p = \infty$, (iii) $r = p = \infty$
(2) Induction Step $(m - 1 \to m)$

(i) Claim :

$$\|\nabla^m v\|_{q'} \leqslant c\|\nabla^{m+1} v\|_p^{m/(m+1)}\|v\|_r^{1-m/(m+1)}$$
$$\text{where} \quad \frac{1}{q'} = \frac{m}{m+1}\frac{1}{p} + \frac{1}{m+1}\frac{1}{r}$$

(ii) Claim :

$$\|\nabla^j v\|_q \leqslant c\|\nabla^{m+1} v\|_p^{j/(m+1)}\|v\|_r^{1-j/(m+1)} \text{ for any } j \in \{1, \cdots, m\}$$
$$\text{where} \quad \frac{1}{q} = \frac{j}{m+1}\frac{1}{p} + (1 - \frac{j}{m+1})\frac{1}{r}$$

References.

[1] R. Racke, *Lectures on Nonlinear Evolution Equations.*