Seminar Handout (Week 2)

Introduction to Wave Equations

D. Kim

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1. Problem

Consider the following initial value problem

$$y_{tt} - \Delta y = 0$$

$$y(t=0) = 0, \quad y_t(t=0) = g$$

where $y = y(t,x)$, $t \geqslant 0$, $x \in \mathbb{R}^n$. Now define the operator $w(t)$ by

$$(w(t)g)(x) := y(t,x).$$

2. Theorem.

Theorem 2.1 *There exists $c = c(n) > 0$ for all $g \in C_0^\infty$ and $t \geqslant 0$:*

(i) $||Dw(t)g||_2 = ||g||_2$
(ii) $||Dw(t)g||_\infty \leqslant c(1+t)^{-\frac{n-1}{2}} ||g||_{n,1}$.

3. Interpolation.

Definition A.0 *Let X, Y be normed vector spaces with norms $||\cdot||_X, ||\cdot||_Y$ such that $X \subset Y$. X is said to **continuously embedded** into Y if there exists a $C \geqslant 0$ such that $||x||_Y \leqslant C||x||_X$ for all $x \in X$.*

Definition A.1 *(X_0, X_1) is called an **interpolation couple** if X_0, X_1 are Banach spaces which are continuously embedded into a topological Hausdorff space.*

Lemma A.2 *Let (X_0, X_1) be an interpolation couple. Then*

(i) *$X_0 \cap X_1$ is a Banach space with norm*

$$||x||_{X_0 \cap X_1} := \max\{||x||_{X_0}, ||x||_{X_1}\},$$

(ii) $X_0 + X_1$ *is a Banach space with norm*

$$||x||_{X_0+X_1} := \inf\{||x_0||_{X_0} + ||x_1||_{X_1} \mid x_0 \in X_0,\ x_1 \in X_1,\ x_0 + x_1 = x\}$$

Definition A.3 *Let* (X_0, X_1) *and* (Y_0, Y_1) *be interpolation couples and let* X, Y *be Banach spaces.*

(i) X *is called an* ***intermediate space*** *between* X_0 *and* X_1 *if* $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$.
(ii) X, Y *are called* ***interpolation spaces*** *for* $(X_0, X_1), (Y_0, Y_1)$ *if* X *is an intermediate space between* X_0 *and* X_1, Y *is an intermediate space between* Y_0 *and* Y_1 *and*

$$T : X_j \to Y_j \text{ is continuous } for\ j = 0, 1 \ \Rightarrow\ T : X \to Y \text{ is continuous.}$$

References

[1] R. Racke, *Lectures on Nonlinear Evolution Equations*