Seminar Handout (Week 3,4)

Linear Symmetric Hyperbolic Systems

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1. $L^p - L^q$ Decay Estimates for the Linear Wave Eqs.

Problems.

$$y_{tt} - \Delta y = 0,$$
$$y(t=0) = 0, y_t(t=0) = g,$$
$$(w(t)g)(x) := y(t,x).$$

Theorem 2.2. There exists $c = c(n) > 0$ for any $g \in W^{n,1},\ t \geqslant 0$:
(i) $||Dw(t)g||_2 = ||g||_2$
(ii) $||Dw(t)g||_\infty \leqslant c(1+t)^{-\frac{n-1}{2}}||g||_{n,1}$

Theorem 2.3. Let $2 \leqslant q \leqslant \infty,\ 1/p + 1/q = 1,\ N_p > n(1-2/q)$. Then there exists $c = c(q,n) > 0$ for any $g \in W^{N_p,p},\ \ t \geqslant 0$:

$$||Dw(t)g||_q \leqslant c(1+t)^{-\frac{n-1}{2}(1-\frac{2}{q})}||g||_{N_p,p}.$$

2. Linear Symmetric Hyperbolic Systems.

Gronwall's Inequality. Assume $\alpha, \beta, u \in C^0([a,b]),\ \beta \leqslant 0$ on $[a,b]$. Let $u(t) \leqslant \alpha(t) + \int_a^t \beta(s)u(s)ds$ for all $t \in [a,b]$. Then
(1) $u(t) \leqslant \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr}ds$.
(2) If α is non-decreasing, then $u(t) \leqslant \alpha(t)e^{\int_a^t \beta(r)dr}$
for all $t \in [a,b]$.

Problems. Let $u = u(t,x) = (u_1, \cdots, u_N)(t,x),\ t \leqslant 0,\ x \in \mathbb{R}^n,\ N \in \mathbb{N}$, and define the linear operator L by

$$Lu := A^0(t,x)\partial_t u + \sum_{j=1}^n A^j(t,x)\partial_j u + B(t,x)u.$$

Here $A^0, A^1, \cdots, A^n, B$ are complex $N \times N$ matrices depending on t, x. $A^j,\ 0 \leqslant j \leqslant n$, is assumed to be hermitian and A^0 is assumed to be positive definite, uniformly with respect to t and x.
With these assumptions L is a symmetric hyperbolic differential operator and the system of equations

$$Lu = f,\ u(t=0) = u_0$$

is a symmetric hyperbolic system with data

$$f = f(t,x),\ u_0 = u_0(x).$$

Energy Estimates. Assume

$$A^0, A^1, \cdots, A^n \in C_b^1,\ B \in C_b^0.$$

And define

$$a_0 := \min_{v,t,x;|v|=1} A^0(t,x)v \cdot \bar{v} > 0,$$

$$a_1 := \max_{v,t,x,j;|v|=1} |A^j(t,x)v \cdot \bar{v}| > 0,$$

$$\alpha := \frac{a_0}{na_1},$$

$$K^\alpha := \{(t,x) | x \in K_t, 0 \leqslant t \leqslant t_0\},$$

$$K_t := B(0, \frac{T_0 - t}{\alpha}) \subset \mathbb{R}^n,\ 0 \leqslant t \leqslant T_0,$$

$$|u(t)|_{K_t} := \left(\int_{K_t} A^0(t,x)u(t,x) \cdot \overline{u(t,x)} dx \right)^{1/2}.$$

Theorem 3.1. Let $K^\alpha = K^\alpha(t_0)$ and let $u \in C^1(K^\alpha(T_0))$ be a solution to

$$Lu = f \in C^0(K^\alpha(T_0)),\ \ u(t=0) = u_0 \in C^0(K_0).$$

Then

$$\exists c = c(||(A^0, \partial_t A^0, \partial_1 A^1, \cdots, \partial_n A^n, B)||_{C^0(K^\alpha(T_0))}),\ \ \forall t \in [0, T_0]:$$

$$|u(t)|_{K_t} \leqslant c \left[|u_0|_{K_0} + \left(\int_0^{T_0} |f(r)|_{K_r}^2 dr \right)^{\frac{1}{2}} \right] e^{ct}.$$

References.

[1] R. Racke, *Lectures on Nonlinear Evolution Equations.*