Seminar Handout (Week 9)

Some Inequalities

D. Kim

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1. Theorems

Theorem 4.4. Let $1 \leqslant r, p \leqslant \infty,\ m \in \mathbb{N}$. Then there is a constant $c > 0$ such that for all $w \in W^{m,p} \cap L^r$ the inequality

$$||\nabla^j w||_q \leqslant c||\nabla^m w||_p^{j/m}||w||_r^{1-j/m}$$

holds, where $j \in \{0, 1, \cdots, m\}$ and

$$\frac{1}{q} = \frac{j}{mp} + (1 - \frac{j}{m})\frac{1}{r}.$$

2. Lemmas

Lemma 4.5. Let $-\infty < a < b < \infty,\ f \in C^2([a,b]),\ 1 \leqslant p, r \leqslant \infty,\ 2/q = 1/r + 1/p$, and $q \neq \infty$. Then

$$||f'||_{L^q((a,b))} \leqslant 18^q \left((b-a)^{1+q-q/p}||f''||_{L^p((a,b))}^q + (b-a)^{-(1+q-q/p)}||f||_{L^r((a,b))}^q\right)$$

Lemma 4.6. Let $1 \leqslant p, r \leqslant \infty,\ f \in C^2(\mathbb{R}) \cap L^r(\mathbb{R}),\ f'' \in L^p(\mathbb{R}),\ 2/q = 1/p + 1/r$. Then we have $f' \in L^q(\mathbb{R})$ and there is a constant $c = c(q) > 0$ such that

$$||f'||_q \leqslant c||f''||_p^{1/2}||f||_r^{1/2}.$$

Proof.
(1) $q \neq \infty$
  (i) Claim : $\sum_{i=1}^{k'} ||f''||_{L^p((a_{i-1},a_i))}^{q/2}||f||_{L^r((a_{i-1},a_i))}^{q/2} \leqslant ||f''||_p^{q/2}||f||_r^{q/2}$.
    (α) $r \neq \infty, p \neq \infty$.
    (β) $r = \infty$ or $p = \infty$.
  (ii) Claim : $R \to 0$ as $k \to \infty$.
    (α) $p = \infty$.
    (β) $p \neq \infty$.
      (a) $p/q > 1$.
      (b) $p/q \leqslant 1$.
(2) $q = \infty$

References.

[1] R. Racke, *Lectures on Nonlinear Evolution Equations.*