解析学特論 (240792, 砂川秀明)

R E P O R T

Donghyun Kim (24A12035)

E-mail : kdh743@gmail.com

2013. 2. 4.

1. Introduction

This report consists of two sections. In the first section, we will prove the global regularity of elliptic partial differential equations. In particular we will extend a result in Hess and Kato [2] to the L^p analogue by applying the theory of pseudo differential operators. In the second section, we will give solutions to some selected exercises which were given in the class.

We introduce some notations here. Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{N}_0^n = \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ and $\alpha, \beta, \gamma, \cdots$ be multi-indexes. And let $D_{x_j} = \frac{1}{i}\partial_{x_j}$, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.
Denote L^p as the Lebesgue space, $H^{s,p}$ as the Sobolev space. For the simplicity, let $H^s = H^{s,2}$. Let $\mathcal{S}$ be the Schwartz space with $|\varphi|_{l,\mathcal{S}} = \max_{|\alpha|+k\leqslant l} \sup_{x\in\mathbb{R}^n} \langle x\rangle^k |\partial_x^\alpha \varphi(x)|$ and denote $\mathcal{S}'$ as the space of tempered distributions where $\langle\ \cdot\ \rangle = \sqrt{1+|\cdot|^2}$. For the simplicity, let $X = X(\mathbb{R}^n)$ for some function spaces X.
And we use the Fourier transform of f which is given by $(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int f e^{-ix\cdot\xi}\, dx$ and the inverse $\mathcal{F}^{-1}f = \int f e^{ix\cdot\xi}\, \frac{d\xi}{(2\pi)^n} = \int f e^{ix\cdot\xi}\, \bar{d}\xi$.
Denote Os - $\int\cdots$ as various oscillatory integrals. Also denote S^m as the symbol class of order m. Especially denote $S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m$, $S^\infty = \bigcup_{m\in\mathbb{R}} S^m$ respectively.
Let P (or $Q, R, \cdots$) $= \mathrm{Op}(p)$ (or $\mathrm{Op}(q), \mathrm{Op}(r), \cdots$) be pseudo differential operators with symbols $p, q, r, \cdots$. Also denote their symbols as $\sigma(P), \sigma(Q) \cdots = p, q \cdots$.

2. Global regularity of elliptic PDEs

In this section, we will extend a result in Hess and Kato [2] to the L^p analogue by using the theory of pseudo differential operators.

Definition. *We say $P = P(x, D_x) \in \mathrm{Op}S^m$ is elliptic if there exists $C > 0$ and $R > 0$ such that $|p(x,\xi)| \geqslant C|\xi|^m$ for all $x \in \mathbb{R}^n$, $|\xi| \geqslant R$.*

Lemma 1. *If $P \in \mathrm{Op}S^m$ is elliptic. Then the following holds.*
(i) *$\exists Q \in \mathrm{Op}S^{-m}$, $\exists R \in \mathrm{Op}S^{-\infty}$ such that $QP = I + R$.*
(ii) *$\exists Q \in \mathrm{Op}S^{-m}$, $\exists R' \in \mathrm{Op}S^{-\infty}$ such that $PQ = I + R'$.*

Proof. (i) Done in the class.
(ii) Our claim is finding a sequence $q_j \in S^{-m-j}$ for $j \in \mathbb{N}_0$ in the form $\sigma(Q) \sim \sum_{j=0}^\infty q_j$ so that $\sigma(PQ) - 1 \in S^{-\infty}$. Define $\chi \in C^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 0$ $(|\xi| \leqslant r)$, $\chi(\xi) = 1$ $(|\xi| \geqslant 2r)$.

And let $q_0(x,\xi) := \frac{\chi(\xi)}{p(x,\xi)}$ so that $q_0 \in S^{-m}$ and $p(x,\xi)q_0(x,\xi) - 1 = \chi(\xi) - 1 \in S^{-\infty}$. Now define q_ℓ as

$$q_\ell(x,\xi) := -\left(\sum_{j+|\alpha|=\ell,\alpha\neq 0} \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q_j(x,\xi)\right) q_0(x,\xi)$$

for $\ell \in \mathbb{N}$ so that $q_\ell \in S^{m-\ell}$. So there exists $q \in S^{-m}$ such that $q \sim \sum_{j=0}^{\infty} q_j$ or $q - \sum_{j=0}^{L} q_j \in S^{-m-L-1}$ for all $L \in \mathbb{N}_0$. Then we have

$$\sigma(PQ) - \sum_{|\alpha|\leqslant N} \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) \in S^{-N-1} \quad (\forall N \in \mathbb{N}_0). \tag{2.1}$$

Note that $D_x^\alpha q(x,\xi) - \sum_{j=0}^{L} D_x^\alpha q_j(x,\xi) \in S^{-L-1-|\alpha|-m}$. If we put $L = N - |\alpha|$ then

$$D_x^\alpha q(x,\xi) - \sum_{j=0}^{N-|\alpha|} D_x^\alpha q_j(x,\xi) \in S^{-1-N-m}.$$

Multiplying $\frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi)$ both sides and taking summation over $|\alpha| \leqslant N$ we get

$$\sum_{|\alpha|\leqslant N} \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) - \sum_{j+|\alpha|\leqslant N} \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q_j(x,\xi) \in S^{-1-N}. \tag{2.2}$$

Here we have

$$\begin{aligned}
\sum_{j+|\alpha|\leqslant N} \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q_j(x,\xi) &= \sum_{\ell=0}^{N} \sum_{j+|\alpha|=\ell} \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q_j(x,\xi) \\
&= pq_0 + \sum_{\ell=1}^{N} \Bigg(pq_\ell + \underbrace{\sum_{j+|\alpha|=\ell,\alpha\neq 0} \frac{1}{\alpha!}\partial_\xi^\alpha p D_x^\alpha q_j}_{=:a_\ell(x,\xi)} \Bigg) \\
&= pq_0 + \sum_{\ell=1}^{N} (-a_\ell p q_0 + a_\ell) \\
&= 1 + (pq_0 - 1)(1 - \sum_{\ell=1}^{N} a_\ell) \equiv 1 \mod S^{-\infty}.
\end{aligned} \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), we get $\sigma(PQ) - 1 \in S^{-N-1}$ for $N \in \mathbb{N}$ so that $\sigma(PQ) \equiv 1 \mod S^{-\infty}$ which proves (ii). □

Theorem 1. *Let $P(x,D_x) = \sum_{|\alpha|\leqslant m} a_\alpha(x) D_x^\alpha$ be a linear partial differential operator of order m such that $\sup_{x\in\mathbb{R}^n} |(D_x^\beta a_\alpha)(x)| < \infty$ for $|\alpha| \leqslant m$ and any β so that $P(x,D_x)$ is a pseudo differential operator with symbol $p(x,\xi) = \sum_{|\alpha|\leqslant m} a_\alpha(x)\xi^\alpha \in S^m$. Let $x_0 \in \mathbb{R}^n$ and suppose that there exists $C_1, C_2 > 0$ such that*

$$\left|\sum_{|\alpha|=m} a_\alpha(x_0)\xi^\alpha\right| \geqslant C_1|\xi|^m \qquad (2.4), \qquad \left|\sum_{|\alpha|=m} (a_\alpha(x) - a_\alpha(x_0))\xi^\alpha\right| \leqslant C_2|\xi|^m \qquad (2.5)$$

for all $(x,\xi) \in \mathbb{R}^{2n}$, $C_2 < C_1$. Then if $u \in H^{s,p}$, $Pu = f$ and $f \in H^{s,p}$, then $u \in H^{s+m,p}$.

For the proof of the **Theorem 1**, we introduce the following lemma:

Lemma 2. *Let $P(x,D_x)=\sum_{|\alpha|\leqslant m}a_\alpha(x)D_x^\alpha$ be a linear partial differential operator satisfying the hypotheses of* **Theorem 1**. *Then there exist $C,R>0$ such that*

$$|p(x,\xi)| \geqslant C|\xi|^m \qquad \textit{for} \quad |\xi|\geqslant R.$$

Proof. Using (2.4) and (2.5), for any $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$, we have

$$\left|\sum_{|\alpha|=m}a_\alpha(x)\xi^\alpha\right| \geqslant \left|\sum_{|\alpha|=m}a_\alpha(x_0)\xi^\alpha\right| - \left|\sum_{|\alpha|=m}(a_\alpha(x)-a_\alpha(x_0))\xi^\alpha\right| \geqslant (C_1-C_2)|\xi|^m \geqslant C|\xi|^m.$$

Therefore we get

$$\left|\sum_{|\alpha|\leqslant m}a_\alpha(x)\xi^\alpha\right| \geqslant \left|\sum_{|\alpha|=m}a_\alpha(x)\xi^\alpha\right| - \left|\sum_{|\alpha|<m}a_\alpha(x)\xi^\alpha\right| \geqslant C|\xi|^m - C'|\xi|^{m-1} \geqslant C|\xi|^m$$

for sufficiently large $|\xi|\geqslant R$. □

Proof of Theorem 1. By the **Lemma 2**, $P(x,D_x)$ is elliptic with symbol $p(x,\xi)\in S^m$. Therefore by the **Lemma 1**, there exists $Q\in \mathrm{Op}S^{-m}$ $(q\in S^{-m})$ and $R\in \mathrm{Op}S^{-\infty}\subset \mathrm{Op}S^{-m}$ $(r\in S^{-m})$ such that $QP=I+R$. Therefore we get $QPu=(I+R)u=u+Ru$ or $Qf=u+Ru$ since $Pu=f$ by the assumption. Thus we can write as

$$u=Qf-Ru. \tag{2.6}$$

Observe that $Qf\in H^{s+m,p}$ because $q\in S^{-m}$ and $f\in H^{s,p}$. Also observe that $Ru\in H^{s+m,p}$ since $r\in S^{-m}$ and $u\in H^{s,p}$. Finally by (2.6), we have $u\in H^{s+m,p}$. □

3. Exercises

Exercise 1. Let $p\geqslant 1$. Show that there is no u satisfying both (i) $u\in L^p(\mathbb{R}^n)$ and (ii) $\delta[\varphi]=\langle u,\varphi\rangle$ $(\forall\varphi\in\mathcal{S})$ where $\delta:\mathcal{S}\ni\varphi\mapsto\varphi(0)\in\mathbb{C}$.

Proof. Note that for any $1\leqslant r\leqslant\infty$, $\mathcal{S}(\mathbb{R}^n)\subset L^r(\mathbb{R}^n)$. Let $p>1$ and $q=\left(1-\frac{1}{p}\right)^{-1}$ so that $\frac{1}{p}+\frac{1}{q}=1$ and $1<q<\infty$. Fix $\varphi\in\mathcal{S}$ and let $\varphi_k=k^{\frac{n}{q}}\varphi(kx)$ so that $\varphi_k\in\mathcal{S}$ for all $k\in\mathbb{N}$. Then we have

$$\lim_{k\to\infty}|\delta[\varphi_k]| = \lim_{k\to\infty}|\varphi_k(0)| = \infty$$

because $1<q<\infty$. But by the Hölder's inequality,

$$\begin{aligned}|\langle u,\varphi_k\rangle| &= \left|\int_{\mathbb{R}^n}u(x)\varphi_k(x)\,dx\right| \leqslant \|u\|_{L^p}\|\varphi_k\|_{L^q} = \|u\|_{L^p}\left(\int_{\mathbb{R}^n}k^n|\varphi^q(kx)|\,dx\right)^{1/q}\\ &= \|u\|_{L^p}\left(\int_{\mathbb{R}^n}|\varphi^q(y)|\,dy\right)^{1/q} = \|u\|_{L^p}\|\varphi\|_{L^q}.\end{aligned}$$

for all $k\in\mathbb{N}$. Thus if $u\in L^p$ then (ii) fails for all $p>1$.
Now let $p=1$. Fix $\varphi\in C_0^\infty\subset\mathcal{S}$ and let $\varphi_k=\varphi(kx)$, $\varphi(0)=1$. Then $\lim_{k\to\infty}|\delta[\varphi_k]|=\lim_{k\to\infty}|\varphi_k(0)|=1$. But because $\varphi_k=\varphi(kx)\in C_0^\infty$, there exists $C_1>0, C_2>0$ such that $|\varphi(kx)|\leqslant C_1\chi_{|x|\leqslant\frac{C_2}{k}}$. Thus we have

$$|\langle u,\varphi_k\rangle| \leqslant \int_{\mathbb{R}^n}|u(x)\varphi(kx)|\,dx \leqslant \int_{\mathbb{R}^n}|u(x)|C_1\chi_{|x|\leqslant\frac{C_2}{k}}\,dx = C_1\int_{|x|\leqslant\frac{C_2}{k}}|u(x)|\,dx$$

so that $|\langle u, \varphi_k\rangle| \leqslant C_1 \int_{|x|\leqslant \frac{C_2}{k}} |u(x)|\, dx \to 0$ as $k \to \infty$ if $u \in L^1$. Thus if $u \in L^1$ then (ii) fails. This completes the proof. □

Exercise 2. Let $u \in \mathcal{S}'$ and let

(i) $\langle D_x^\alpha u, \varphi\rangle := (-1)^{|\alpha|}\langle u, D_x^\alpha \varphi\rangle$,
(ii) $\langle fu, \varphi\rangle := \langle u, f\varphi\rangle$ with f : slowly increasing function,
(iii) $\langle u \circ \tau_h, \varphi\rangle := \langle u, \varphi \circ \tau_{-h}\rangle$ where $\tau_h : x \mapsto x - h$ $(h \in \mathbb{R}^n)$,
(iv) $\langle u \circ d_\lambda, \varphi\rangle := \lambda^{-n}\langle u, \varphi \circ d_{1/\lambda}\rangle$ where $d_\lambda : x \mapsto \lambda x$ $(\lambda > 0)$,
(v) $\langle u^\vee, \varphi\rangle := \langle u, \varphi^\vee\rangle$ where $\cdot^\vee : x \mapsto -x$,
(vi) $\langle u * \psi, \varphi\rangle := \langle u, \varphi * \psi^\vee\rangle$ where $*$: convolution.

Then show that (i)∼(vi) define the elements of $\mathcal{S}'$ (Note that if $\varphi, \psi \in \mathcal{S}$, then $D_x^\alpha\varphi, \alpha\varphi, \varphi \circ \tau_{-h}, \varphi \circ d_{1/\lambda}, \varphi^\vee, \varphi * \psi^\vee \in \mathcal{S}$).

Proof. For the simplicity, denote $T_u[\varphi] = \langle T_u, \varphi\rangle$. The linearity is almost trivial, for example, if $a, b \in \mathbb{C}$ then by the linearity of u,

$$\begin{aligned}(u * \psi)[a\varphi_1 + b\varphi_2] &= \langle u, (a\varphi_1 + b\varphi_2) * \psi^\vee\rangle = \langle u, a(\varphi_1 * \psi^\vee) + b(\varphi_2 * \psi^\vee\rangle \\ &= u[a(\varphi_1 * \psi^\vee) + b(\varphi_2 * \psi^\vee)] = au[\varphi_1 * \psi^\vee] + bu[\varphi_2 * \psi^\vee] \\ &= a(u * \psi)[\varphi_1] + b(u * \psi)[\varphi_2]\end{aligned}$$

and the other cases are similar. Now we only prove the continuity of (i)∼(vi).

Let $\varphi_j \longrightarrow \varphi$ in $\mathcal{S}$ as $j \to \infty$.

(i) For any $\ell \in \mathbb{N}_0$,

$$|D_x^\alpha\varphi_j - D_x^\alpha\varphi|_{\ell,\mathcal{S}} = |D_x^\alpha(\varphi_j - \varphi)|_{\ell,\mathcal{S}} \leqslant |\varphi_j - \varphi|_{\ell+|\alpha|,\mathcal{S}} \longrightarrow 0 \ (j \to \infty)$$

which means that $D_x^\alpha\varphi_j \longrightarrow D_x^\alpha\varphi$ in $\mathcal{S}$. So by the continuity of u, $(-1)^{|\alpha|}u[D_x^\alpha\varphi_j] \longrightarrow (-1)^{|\alpha|}u[D_x^\alpha\varphi]$ which means that $(D_x^\alpha u)[\varphi_j] \longrightarrow (D_x^\alpha u)[\varphi]$ as $j \to \infty$ so that $D_x^\alpha u \in \mathcal{S}'$.

(ii) Because f is slowly increasing function, for any $\beta \in \mathbb{N}_0^n$, there exists $C_\beta > 0$ and $M_\beta > 0$ such that $|\partial_x^\beta f(x)| \leqslant C_\beta\langle x\rangle^{M_\beta}$. Thus by the Leibniz rule, we have

$$\begin{aligned}\langle x\rangle^k|\partial_x^\alpha(f(\varphi_j - \varphi))| &\leqslant \langle x\rangle^k C \sum_{|\beta|+|\gamma|=|\alpha|} |(\partial_x^\beta f)(\partial_x^\gamma(\varphi_j - \varphi))| \\ &\leqslant C\langle x\rangle^{k+M_\beta} \sum_{|\gamma|\leqslant|\alpha|} |\partial_x^\gamma(\varphi_j - \varphi)| \leqslant C|\varphi_j - \varphi|_{k+M_\beta+|\alpha|,\mathcal{S}}.\end{aligned}$$

So for any $\ell \in \mathbb{N}_0$, $|f\varphi_j - f\varphi|_{\ell,\mathcal{S}} \longrightarrow 0$ as $j \to \infty$. Thus the continuity of u yields $(fu)[\varphi_j] \longrightarrow (fu)[\varphi]$ so that $fu \in \mathcal{S}'$.

(iii), (iv), (v) For any $\ell \in \mathbb{N}_0$,

$$\left.\begin{aligned}&|\varphi_j \circ \tau_{-h} - \varphi \circ \tau_{-h}|_{\ell,\mathcal{S}} = |\varphi_j(x+h) - \varphi(x+h)|_{\ell,\mathcal{S}} \\ &|\varphi_j \circ d_{1/\lambda} - \varphi \circ d_{1/\lambda}|_{\ell,\mathcal{S}} = \left|\varphi_j\left(\tfrac{x}{\lambda}\right) - \varphi\left(\tfrac{x}{\lambda}\right)\right|_{\ell,\mathcal{S}} \\ &|\varphi_j^\vee - \varphi^\vee|_{\ell,\mathcal{S}} = |\varphi_j(-x) - \varphi(-x)|_{\ell,\mathcal{S}}\end{aligned}\right\} = |\varphi_j - \varphi|_{\ell,\mathcal{S}} \longrightarrow 0$$

as $j \to \infty$. So the continuity of u yields $u \circ \tau_h[\varphi_j] \longrightarrow u \circ \tau_h[\varphi]$, $u \circ d_\lambda[\varphi_j] \longrightarrow u \circ d_\lambda[\varphi]$, $u^\vee[\varphi_j] \longrightarrow u^\vee[\varphi]$ so that $u \circ \tau_h, u \circ d_\lambda, u^\vee \in \mathcal{S}'$.

(vi) Note that for any $\ell \in \mathbb{N}_0$ and $f, g \in \mathcal{S}$, $|f * g|_{\ell,\mathcal{S}} \leqslant C|f|_{\ell+n+1,\mathcal{S}}|g|_{\ell,\mathcal{S}}$. Thus we get

$$|\varphi_j * \psi^\vee - \varphi * \psi^\vee|_{\ell,\mathcal{S}} \;=\; |(\varphi_j - \varphi) * \psi^\vee|_{\ell,\mathcal{S}} \;\leqslant\; C|\varphi_j - \varphi|_{\ell+n+1,\mathcal{S}}|\psi^\vee|_{\ell,\mathcal{S}} \;\longrightarrow 0$$

as $j \to \infty$ and the continuity of u yields $(u * \psi)[\varphi_j] \longrightarrow (u * \psi)[\varphi]$ so that $u * \psi \in \mathcal{S}'$. □

Exercise 3. Let $k \in \mathbb{N}_0$. Prove that
(i) $\mathcal{S}$ is dense in H^k. (ii) H^k is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{(k)}$.
(iii) There exist $C_1 > 0$ and $C_2 > 0$ such that $C_1\|u\|_{(k)}^2 \leqslant \int_{\mathbb{R}^n} \langle \xi \rangle^{2k} |\hat{u}(\xi)|^2 \, d\xi \leqslant C_2\|u\|_{(k)}^2$ for any $u \in H^k$ (In this exercise, $k = 0, 1, 2, \cdots$ for the moment and let D_x^α be defined by (i) in Exercise 2). Clearly $\langle \cdot, \cdot \rangle_{(k)}$ is an inner product so that $\|u\|_{(k)}^2 := \langle u, u \rangle_{(k)} = \sum_{|\alpha| \leqslant k} \|D_x^\alpha u\|_{L^2}^2$ is a norm.

Proof. (i) We are enough to prove that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n)$ because $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. Let $u \in H^k$. Define Friedrichs mollifier $j_\varepsilon(x) = \varepsilon^{-n} j(x/\varepsilon)$ for $\varepsilon > 0$ where $j \in C_0^\infty$ with $j(x) = 0$ $(|x| \geqslant 1)$, $j(x) \geqslant 0$, $\int_{\mathbb{R}^n} j(x) \, dx = 1$. And let $J_\varepsilon u = j * u$. Then we have $J_\varepsilon u \in C^\infty \cap H^k$ and $\lim_{\varepsilon \downarrow 0} \|u - J_\varepsilon u\|_{(k)} = 0$ which means that C_0^∞ is dense in H^k.

(ii) We will show the completeness. Let $\{u_m\}$ be a Cauchy sequence in H^k. Then for each $|\alpha| \leqslant k$, $\{D_x^\alpha u_m\}$ is a Cauchy sequence in L^2. Then by the completeness of L^2, there exists $u_\alpha \in L^2$ such that $D_x^\alpha u_m \longrightarrow u_\alpha$ in L^2. In particular $u_m \longrightarrow u_{(0,\cdots,0)} =: u_0$ in L^2. Note that $\lim_{m\to\infty} \langle D_x^\alpha u_m, \varphi \rangle = \langle u_\alpha, \varphi \rangle$ for any $|\alpha| \leqslant k$ and $\varphi \in \mathcal{S}$ $(\subset L^2)$ since

$$|\langle D_x^\alpha u_m, \varphi \rangle - \langle u_\alpha, \varphi \rangle| \;=\; |\langle D_x^\alpha u_m - u_\alpha, \varphi \rangle| \;\leqslant\; \|D_x^\alpha u_m - u_\alpha\|_{L^2} \|\varphi\|_{L^2} \;\longrightarrow 0.$$

Now fix $\varphi \in \mathcal{S}$ then we get

$$\langle D_x^\alpha u_0, \varphi \rangle \;=\; (-1)^{|\alpha|} \langle u_0, D_x^\alpha \varphi \rangle \;=\; (-1)^{|\alpha|} \lim_{m\to\infty} \langle u_m, D_x^\alpha \varphi \rangle \;=\; \lim_{m\to\infty} \langle D_x^\alpha u_m, \varphi \rangle \;=\; \langle u_\alpha, \varphi \rangle$$

which means that $u_\alpha = D_x^\alpha u_0$ so that $D_x^\alpha u_m \longrightarrow D_x^\alpha u_0$ in L^2 or $u_m \longrightarrow u$ in H^k.

(iii) Note that $|\mathcal{F} D_x^\alpha u|^2 = |\xi^\alpha \mathcal{F} u|^2 \leqslant C\langle \xi \rangle^{2k} |\mathcal{F} u|^2$ for $|\alpha| \leqslant k$. Therefore by the Plancherel's theorem,

$$\|u\|_{(k)}^2 \;=\; \sum_{|\alpha| \leqslant k} \|D_x^\alpha u\|_{L^2}^2 \;=\; C \sum_{|\alpha| \leqslant k} \|\mathcal{F} D_x^\alpha u\|_{L^2}^2 \;\leqslant\; C\|\langle \xi \rangle^k \mathcal{F} u\|_{L^2}^2.$$

Also using the property that $\xi^\alpha \mathcal{F} u = \mathcal{F} D_x^\alpha u$ and the Plancherel's theorem, we have

$$\begin{aligned} \|\langle \xi \rangle^k \mathcal{F} u\|_{L^2}^2 \;&\leqslant\; \sum_{|\alpha| \leqslant k} \|\langle \xi \rangle^{|\alpha|} \mathcal{F} u\|_{L^2}^2 \;\leqslant\; C \sum_{|\alpha| \leqslant k} \|\xi^\alpha \mathcal{F} u\|_{L^2}^2 \\ &=\; C \sum_{|\alpha| \leqslant k} \|\mathcal{F} D_x^\alpha u\|_{L^2}^2 \;=\; C \sum_{|\alpha| \leqslant k} \|D_x^\alpha u\|_{L^2}^2 \;=\; C\|u\|_{(k)}^2. \end{aligned}$$

Thus the norm $\sum_{|\alpha| \leqslant k} \|D_x^\alpha u\|_{L^2}^2$ and $\|\langle \xi \rangle^k \mathcal{F} u\|_{L^2}^2$ are equivalent. □

Exercise 4. Let $s = k + r$ with $k \in \mathbb{Z}$ and $r \in (0, 1)$. Then prove that

$$u \in H^s(\mathbb{R}^n) \;\Leftrightarrow\; \|u\|_{L^2}^2 + \sum_{|\alpha| = k} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{n+2r}} \, dxdy < \infty.$$

Proof. For the simplicity let

$$K := \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{n+2r}}\,dxdy \;=\; \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\partial^\alpha u(x+y)-\partial^\alpha u(y)|^2}{|x|^{n+2r}}\,dxdy$$

and

$$c_r(\xi) := \int_{\mathbb{R}^n} \frac{|e^{ix\cdot\xi}-1|^2}{|x|^{n+2r}}\,dx.$$

Note that by the Plancherel's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\partial^\alpha u(x+y)-\partial^\alpha u(y)|^2\,dy &= C\|\mathcal{F}\partial^\alpha u(x+y)-\mathcal{F}\partial^\alpha u(y)\|_{L^2}^2\\
&= C\|\xi^\alpha \mathcal{F}u(x+y)-\xi^\alpha\mathcal{F}u(y)\|_{L^2}^2\\
&= C\|\xi^\alpha e^{ix\cdot\xi}\hat{u}(\xi)-\xi^\alpha\hat{u}(\xi)\|_{L^2}^2\\
&= C\||\xi|^{|\alpha|}(e^{ix\cdot\xi}-1)\hat{u}(\xi)\|_{L^2}^2\\
&= C\int_{\mathbb{R}^n}|\xi|^{2|\alpha|}|e^{ix\cdot\xi}-1|^2|\hat{u}(\xi)|^2\,d\xi.
\end{aligned}$$

Now consider $c_r(\xi)$. Note that $c_r(\xi)<\infty$ for any $\xi\in\mathbb{R}^n$. Let R be any rotation matrix about the origin in $\mathbb{R}^n$ so that if we substitute $R^{-1}x=y$ then $x\cdot R\xi = R^{-1}x\cdot\xi = y\cdot\xi,\ |x|=|y|$. Then we have

$$c_r(R\xi) \;=\; \int_{\mathbb{R}^n}\frac{|e^{ix\cdot R\xi}-1|^2}{|x|^{n+2r}}\,dx \;=\; \int_{\mathbb{R}^n}\frac{|e^{iy\cdot\xi}-1|^2}{|y|^{n+2r}}\,dy \;=\; c_r(\xi)$$

and

$$c_r(t\xi) \;=\; \int_{\mathbb{R}^n}\frac{|e^{itx\cdot\xi}-1|^2}{|x|^{n+2r}}\,dx \;=\; t^{2r}\int_{\mathbb{R}^n}\frac{|e^{ix\cdot\xi}-1|^2}{|x|^{n+2r}}\,dx \;=\; t^{2r}c_r(\xi)$$

for $t>0$. Therefore we get

$$c_r(\xi) \;=\; |\xi|^{2r}c_r(\xi/|\xi|) \;=\; |\xi|^{2r}c_r(1) \;=\; C|\xi|^{2r}.$$

Combining these results, we have

$$\begin{aligned}
K &= \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\partial^\alpha u(x+y)-\partial^\alpha u(y)|^2}{|x|^{n+2r}}\,dydx \;=\; C\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\xi|^{2|\alpha|}|e^{ix\cdot\xi}-1|^2|\hat{u}(\xi)|^2}{|x|^{n+2r}}\,d\xi dx\\
&= C\int_{\mathbb{R}^n}|\xi|^{2|\alpha|}c_r(\xi)|\hat{u}(\xi)|^2\,d\xi \;=\; C\int_{\mathbb{R}^n}|\xi|^{2|\alpha|+2r}|\hat{u}(\xi)|^2\,d\xi
\end{aligned}$$

so that

$$\begin{aligned}
\|u\|_{L^2}^2+\sum_{|\alpha|=k}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{n+2r}}\,dxdy &= \|\mathcal{F}u\|_{L^2}^2+C\sum_{|\alpha|=k}\int_{\mathbb{R}^n}|\xi|^{2|\alpha|+2r}|\hat{u}(\xi)|^2\,d\xi\\
&= C\int_{\mathbb{R}^n}(1+|\xi|^{2k+2r})|\hat{u}(\xi)|^2\,d\xi.
\end{aligned}$$

Therefore, since $s=k+r$, we can see that this is equivalent to the norm $\|u\|_{(s)} = \|\langle\xi\rangle^s\hat{u}\|_{L^2}^2 = \int_{\mathbb{R}^n}(1+|\xi|^2)^s|\hat{u}(\xi)|^2\,d\xi$. □

Exercise 5. Let $\delta\in\mathcal{S}'$ be a map which is defined by $\delta:\mathcal{S}\ni\varphi\mapsto\varphi(0)$. Show that

$\mathrm{supp}\delta = \{0\}$.

Proof. Note that for $u \in \mathcal{S}'$, $x_0 \notin \mathrm{supp}u$ means that there exists Ω : nbd. of x_0 such that $(\forall \varphi \in \mathcal{S}$ with $\mathrm{supp}\varphi \subset \Omega,\ \langle u, \varphi\rangle = 0)$.
(i) $0 \in \mathrm{supp}\delta$.
Let $\varepsilon > 0$. Let $B_\varepsilon(0)$ be an open ball in $\mathbb{R}^n$ with the radius ε and the center 0. And let $\varphi \in C_0^\infty\ (\subset \mathcal{S})$ with $\varphi(x) = 0$ if $|x| \geqslant \frac{\varepsilon}{2}$ and $\varphi(0) = 1$ so that $\mathrm{supp}\varphi \subset B_\varepsilon(0)$. This means that for any Ω : nbd. of 0, there exists $\varphi \in \mathcal{S}$ with $\mathrm{supp}\varphi \subset \Omega$ and $\langle \delta, \varphi\rangle = \varphi(0) = 1 \neq 0$.
(ii) $\forall x \in \mathbb{R}^n \setminus \{0\},\ x \notin \mathrm{supp}\delta$.
Let $\Omega = B_{x/2}(x)$. Then for any $\varphi \in \mathcal{S}$ with $\mathrm{supp}\varphi \subset \Omega$, $\langle \delta, \varphi\rangle = \varphi(0) = 0$. □

Exercise 6. Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ with $\chi(0) = 1$. Prove that
(i) $\chi(\varepsilon x) \longrightarrow 1$ as $\varepsilon \downarrow 0$ (compact convergence).
(ii) $|\alpha| \geqslant 1 \ \Rightarrow\ \partial_x^\alpha(\chi(\varepsilon x)) \longrightarrow 0$ as $\varepsilon \downarrow 0$ (uniform convergence).
(iii) $\forall \alpha \in \mathbb{N}_0^n,\ \exists C_\alpha$ (indep. of ε) > 0 such that $|\partial_x^\alpha(\chi(\varepsilon x))| \leqslant C_\alpha \langle x\rangle^{-|\alpha|}$.

Proof.
(i) Let $\varepsilon < 1$ and let K be any compact set in $\mathbb{R}^n$. Then we have

$$\begin{aligned} |\chi(\varepsilon x) - 1| &= |\chi(\varepsilon x) - \chi(0)| = \left| \int_0^\varepsilon \partial_t \chi(tx)\, dt \right| \leqslant \int_0^\varepsilon |\nabla\chi(tx) \cdot x|\, dt \\ &\leqslant \varepsilon \sup_{0\leqslant t\leqslant 1,\ x\in K} |\nabla\chi(tx)||x| \leqslant C\varepsilon \end{aligned}$$

because $\chi \in \mathcal{S} \subset C^1$ so that $\chi(\varepsilon x) \longrightarrow 1$ as $\varepsilon \downarrow 0$ compactly.
(ii) Note that since $\lim_{|x|\to\infty} |\partial_x^\alpha \chi(x)| = 0$, there exists $M > 0$ such that if $|x| \geqslant M$ then $|\partial_x^\alpha \chi(x)| < 1$. Therefore setting $C = 1 + \max_{|x|\leqslant M} |\chi(x)|$, we get

$$|\partial_x^\alpha(\chi(\varepsilon x))| = \varepsilon^{|\alpha|} |(\partial_x^\alpha \chi)(\varepsilon x)| \leqslant C\varepsilon^{|\alpha|}.$$

Therefore if $|\alpha| \geqslant 1$, $\partial_x^\alpha(\chi(\varepsilon x)) \longrightarrow 0$ as $\varepsilon \downarrow 0$ uniformly.
(iii) Note that since $\lim_{|x|\to\infty} |x^\alpha \partial_x^\alpha \chi(x)| = 0$, by the similar way we have $|x|^{|\alpha|} |\partial_x^\alpha \chi(x)| \leqslant C$. Because $\varepsilon^{|\alpha|} = |x|^{-|\alpha|} |\varepsilon x|^{|\alpha|}$, we have

$$|\partial_x^\alpha(\chi(\varepsilon x))| = \varepsilon^{|\alpha|} |(\partial_x^\alpha \chi)(\varepsilon x)| = |x|^{-|\alpha|} |\varepsilon x|^{|\alpha|} |\partial_x^\alpha \chi(\varepsilon x)| \leqslant C|x|^{-|\alpha|} \leqslant C\langle x\rangle^{-|\alpha|}.$$ □

Exercise 7. $a \in \mathscr{A}^{m,\tau}$ means that for any $\alpha, \beta \in \mathbb{N}_0^n,\ \exists C_{\alpha,\beta} > 0$ such that $|\partial_\eta^\alpha \partial_y^\beta a(\eta, y)| \leqslant C_{\alpha,\beta} \langle\eta\rangle^m \langle y\rangle^\tau$ $(\eta, y \in \mathbb{R}^n)$. Let $\mathscr{A} = \bigcup_{m\in\mathbb{R}} \bigcup_{\tau\in\mathbb{R}} \mathscr{A}^{m,\tau}$. Let $a \in \mathscr{A}(\mathbb{R}^n \times \mathbb{R}^n)$ then show that

$$\text{(i) Os - } \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta} y^\alpha a(y,\eta)\, dy\bar{d}\eta = \text{Os - } \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta} D_\eta^\alpha a(y,\eta)\, dy\bar{d}\eta$$

$$\text{(ii) Os - } \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta} \eta^\alpha a(y,\eta)\, dy\bar{d}\eta = \text{Os - } \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta} D_y^\alpha a(y,\eta)\, dy\bar{d}\eta$$

where $\bar{d}\eta = d\eta/(2\pi)^n$.

Proof. We will use the fact that for $a_j \in \mathscr{A}$ satisfying $\{a_j\}$ is bounded in $\mathscr{A}$ and $\exists a \in \mathscr{A}$ such that $a_j \longrightarrow a$ compactly, then

$$\lim_{j\to\infty} \text{Os - } \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta} a_j(y,\eta) dy\bar{d}\eta = \text{Os - } \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta} a(y,\eta) dy\bar{d}\eta.$$

(i) Because $y^\alpha e^{-iy\cdot\eta} = (-D_\eta)^\alpha e^{-iy\cdot\eta}$, integration by parts yields

$$\begin{aligned}\text{Os - }\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta}y^\alpha a(y,\eta)\,dy\bar{d}\eta &= \lim_{\varepsilon\downarrow 0}\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta}D_\eta^\alpha(\chi(\varepsilon\eta,\varepsilon y)a(y,\eta))\,dy\bar{d}\eta \\ &= \lim_{\varepsilon\downarrow 0}\text{Os - }\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta}D_\eta^\alpha((\chi(\varepsilon\eta,\varepsilon y)a(y,\eta))\,dy\bar{d}\eta\end{aligned}$$

Let $0<\varepsilon<1$. Note that because $a\in\mathscr{A}$, $\{D_\eta^\alpha(\chi(\varepsilon\eta,\varepsilon y)a(y,\eta))\}$ is bounded in $\mathscr{A}$ and $D_\eta^\alpha(\chi(\varepsilon\eta,\varepsilon y)a(y,\eta))\longrightarrow D_\eta^\alpha a(y,\eta)$ compactly as $\varepsilon\downarrow 0$ on $\mathbb{R}^n\times\mathbb{R}^n$. Therefore we get

$$\lim_{\varepsilon\downarrow 0}\text{Os - }\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta}D_\eta^\alpha((\chi(\varepsilon\eta,\varepsilon y)a(y,\eta))\,dy\bar{d}\eta \;=\; \text{Os - }\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta}D_\eta^\alpha a(y,\eta)\,dy\bar{d}\eta$$

which proves (i).

(ii) This is almost similar to (i). We just change y and η each other.

Exercise 8. Let $p\in S^m$ and fix $\xi\in\mathbb{R}^n$. Let $u(x)=e^{ix\cdot\xi}$. Show that

$$(Pu)(x) \;=\; \text{Os - }\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta}p(x,\eta)e^{i(x+y)\cdot\xi}\,dy\bar{d}\eta \;=\; e^{ix\cdot\xi}p(x,\xi)$$

which implies that $S^m\ni p(x,\xi)\to P(X,D_x)$ is one-to-one.

Proof.

$$\begin{aligned}(Pu)(x) &= \text{Os - }\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot\eta}e^{ix\cdot\xi+iy\cdot\xi}p(x,\eta)\,dy\bar{d}\eta \\ &= e^{ix\cdot\xi}\text{Os - }\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-iy\cdot(\eta-\xi)}p(x,\eta)\,dy\bar{d}\eta \\ &= e^{ix\cdot\xi}\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} p(x,\xi+\eta')\left(\int_{\mathbb{R}^n} e^{-iy\cdot\eta'}\chi(\varepsilon y)\,dy\right)\bar{d}\eta' \\ &= e^{ix\cdot\xi}\lim_{\varepsilon\downarrow 0}\varepsilon^{-n}\int_{\mathbb{R}^n} p(x,\xi+\eta')\hat{\chi}\left(\frac{\eta'}{\varepsilon}\right)\bar{d}\eta' \;=\; e^{ix\cdot\xi}\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} p(x,\xi+\varepsilon\eta'')\hat{\chi}(\eta'')\,\bar{d}\eta'' \\ &= e^{ix\cdot\xi}p(x,\xi)\int_{\mathbb{R}^n}\hat{\chi}(\eta'')\,\bar{d}\eta'' \;=\; e^{ix\cdot\xi}p(x,\xi)\chi(0) \;=\; e^{ix\cdot\xi}p(x,\xi).\end{aligned}$$

□

Exercise 9. Let $p\in S^{m,m'}\cap C^\infty(\mathbb{R}^{4n})$. Here $p\in S^{m,m'}$ means that $\forall\alpha,\beta,\alpha',\beta'\in\mathbb{N}_0^n$, $\exists C=C_{\alpha,\beta,\alpha',\beta'}>0$ such that $|\partial_\xi^\alpha D_x^\beta\partial_{\xi'}^{\alpha'}D_{x'}^{\beta'}p(x,\xi,x',\xi')|\leqslant C\langle\xi\rangle^{m-|\alpha|}\langle\xi'\rangle^{m'-|\alpha'|}$ for all $(x,\xi,x',\xi')\in\mathbb{R}^{4n}$. Define

$$Pu(x) := \text{Os - }\iiiint_{\mathbb{R}^{4n}} e^{-i(y\cdot\xi+y'\cdot\xi')}p(x,\xi,x+y,\xi')u(x+y+y')\,dydy'\bar{d}\xi\bar{d}\xi'$$

for $u\in C_b^\infty(\mathbb{R}^n)$. Then show that

$$Pu(x) \;=\; \int_{\mathbb{R}^n} e^{ix\cdot\xi}\left(\int_{\mathbb{R}^n} e^{-ix'\cdot\xi}\left(\int_{\mathbb{R}^n} e^{ix'\cdot\xi'}p(x,\xi,x',\xi')\hat{u}(\xi')\bar{d}\xi'\right)dx'\right)\bar{d}\xi \qquad (3.0)$$

for $u\in\mathcal{S}(\mathbb{R}^n)$.

Proof. Let $0 < \varepsilon < 1$ and let $\chi(\xi, \xi', y, y') \in \mathcal{S}$ such that $\chi(0,0,0,0) = 1$. Substituting $x' = x + y$, $x'' = x + y + y'$, we have

$$
\begin{aligned}
&Pu(x)\\
&= \text{Os - } \iiiint_{\mathbb{R}^{4n}} e^{-i(y\cdot\xi + y'\cdot\xi')} p(x,\xi,x+y,\xi') u(x+y+y')\, dy dy' \bar{d}\xi \bar{d}\xi'\\
&= \lim_{\varepsilon\downarrow 0} \iiiint_{\mathbb{R}^{4n}} e^{-i(y\cdot\xi + y'\cdot\xi')} \chi(\varepsilon\xi, \varepsilon\xi', \varepsilon y, \varepsilon y') p(x,\xi,x+y,\xi') u(x+y+y')\, dy dy' \bar{d}\xi \bar{d}\xi'\\
&= \lim_{\varepsilon\downarrow 0} \iiiint_{\mathbb{R}^{4n}} e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'} \underbrace{\chi(\varepsilon\xi, \varepsilon\xi', \varepsilon(x'-x), \varepsilon(x''-x)) p(x,\xi,x',\xi') u(x'')}_{=: p_{\varepsilon,u}(x,\xi,x',\xi',x'')}\, dx' dx'' \bar{d}\xi \bar{d}\xi'\\
&= \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(\int_{\mathbb{R}^n} e^{-ix'\cdot\xi} \left(\int_{\mathbb{R}^n} e^{ix'\cdot\xi'} \left(\int_{\mathbb{R}^n} e^{-ix''\cdot\xi'} p_{\varepsilon,u}(x,\xi,x',\xi',x'')\, dx'' \right) \bar{d}\xi' \right) dx' \right) \bar{d}\xi. \quad (3.1)
\end{aligned}
$$

Now take $\ell, \ell' \in \mathbb{N}$ and the even number n_0 satisfying

$$
-2\ell + m < -n, \quad -2\ell' + 2\ell + m' < -n, \quad n_0 > n. \tag{3.2}
$$

For the simplicity, define

$$
q_{\varepsilon,u}(x,\xi,x',\xi') \;=\; \int_{\mathbb{R}^n} e^{-ix''\cdot\xi'} p_{\varepsilon,u}(x,\xi,x',\xi',x'')\, dx'',
$$

$$
r_{\varepsilon,u}(x,\xi,x') \;=\; \int_{\mathbb{R}^n} e^{ix'\cdot\xi'} q_{\varepsilon,u}(x,\xi,x',\xi')\, \bar{d}\xi', \qquad t_{\varepsilon,u}(x,\xi) \;=\; \int_{\mathbb{R}^n} e^{-ix'\cdot\xi} r_{\varepsilon,u}(x,\xi,x')\, dx'
$$

so that

$$
Pu(x) \;=\; \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} e^{ix\cdot\xi} t_{\varepsilon,u}(x,\xi)\, \bar{d}\xi. \tag{3.3}
$$

Now let's investigate the behavior of $q_{\varepsilon,u}$, $r_{\varepsilon,u}$ and $t_{\varepsilon,u}$. Consider $q_{\varepsilon,u}$. Then by the Leibniz rule and using the fact that $e^{-ix''\cdot\xi'} = \langle\xi'\rangle^{-2\ell'} \langle D_{x''}\rangle^{2\ell'} e^{-ix''\cdot\xi'}$, integration by parts yields

$$
\begin{aligned}
D^\alpha_{\xi'} D^\beta_{x'} q_{\varepsilon,u} &= \int_{\mathbb{R}^n} D^\alpha_{\xi'} (e^{-ix''\cdot\xi'} D^\beta_{x'} p_{\varepsilon,u})\, dx''\\
&= \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} \int_{\mathbb{R}^n} e^{-ix''\cdot\xi'} (-x'')^{\alpha_1} D^{\alpha_2}_{\xi'} D^\beta_{x'} p_{\varepsilon,u}\, dx''\\
&= \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} \langle\xi'\rangle^{-2\ell'} \int_{\mathbb{R}^n} e^{-ix''\cdot\xi'} \langle D_{x''}\rangle^{2\ell'} ((-x'')^{\alpha_1} D^{\alpha_2}_{\xi'} D^\beta_{x'} p_{\varepsilon,u})\, dx''.
\end{aligned}
$$

Because $u(x'') \in \mathcal{S}$, $p \in S^{m,m'}$, we get

$$
|D^\alpha_{\xi'} D^\beta_{x'} q_{\varepsilon,u}| \;\leqslant\; C\langle\xi'\rangle^{-2\ell'} \langle\xi\rangle^m \langle\xi'\rangle^{m'-|\alpha_2|} \;\leqslant\; C\langle\xi\rangle^m \langle\xi'\rangle^{m'+|\beta|-2\ell'} \tag{3.4}
$$

provided for $|\alpha| \leqslant n_0$, $|\beta| \leqslant 2\ell$. Similarly by the Leibniz rule, using the fact that $e^{ix'\cdot\xi'} = \langle x'\rangle^{-n_0} \langle D_{\xi'}\rangle^{n_0} e^{ix'\cdot\xi'}$, and the integration by parts yields

$$
\begin{aligned}
D^{\beta'}_{x'} r_{\varepsilon,u} &= \sum_{\beta_1+\beta_2=\beta'} \frac{\beta'!}{\beta_1!\beta_2!} \int_{\mathbb{R}^n} e^{ix'\cdot\xi'} (\xi')^{\beta_1} D^{\beta_2}_{x'} q_{\varepsilon,u}\, \bar{d}\xi'\\
&= \sum_{\beta_1+\beta_2=\beta'} \frac{\beta'!}{\beta_1!\beta_2!} \langle x'\rangle^{-n_0} \int_{\mathbb{R}^n} e^{ix'\cdot\xi'} \langle D_{\xi'}\rangle^{n_0} ((\xi')^{\beta_1} D^{\beta_2}_{x'} q_{\varepsilon,u})\, \bar{d}\xi'.
\end{aligned}
$$

By (3.4), we have $|\langle D_{\xi'}\rangle^{n_0}((\xi')^{\beta_1}D_{x'}^{\beta_2}q_{\varepsilon,u})| \leqslant C\langle\xi\rangle^{m}\langle\xi'\rangle^{m'+|\beta'|-2\ell'}$ provided for $|\beta'| \leqslant 2\ell$. Note that by the relation (3.2), $m'+2\ell-2\ell' < -n$, $\langle\xi'\rangle^{m'+|\beta'|-2\ell'} \leqslant \langle\xi'\rangle^{m'+2\ell-2\ell'} \in L^1(\mathbb{R}^n_{\xi'})$. So

$$|D_{x'}^{\beta'}r_{\varepsilon,u}| \;\leqslant\; C\langle\xi\rangle^{m}\langle x'\rangle^{-n_0}. \tag{3.5}$$

provided for $|\beta'| \leqslant 2\ell$. Using the fact that $e^{-ix'\cdot\xi} = \langle\xi\rangle^{-2\ell}\langle D_{x'}\rangle^{2\ell}e^{-ix'\cdot\xi}$ and (3.5), we get

$$|t_{\varepsilon,u}| \;=\; \langle\xi\rangle^{-2\ell}\left|\int_{\mathbb{R}^n} e^{-ix'\cdot\xi}\langle D_{x'}\rangle^{2\ell}r_{\varepsilon,u}\,dx'\right| \;\leqslant\; C\langle\xi\rangle^{-2\ell}\int_{\mathbb{R}^n}\langle\xi\rangle^{m}\langle x'\rangle^{-n_0}\,dx' \;\leqslant\; C\langle\xi\rangle^{m-2\ell} \tag{3.6}$$

because by the relation (3.2), $n_0 > n$, we have $\langle x'\rangle^{-n_0} \in L^1(\mathbb{R}^n_{x'})$. Also note that by (3.2), $-2\ell+m < -n$, $\langle\xi\rangle^{m-2\ell} \in L^1(\mathbb{R}^n_{\xi})$. Therefore by the Dominated convergence theorem with (3.6), (3.5) with $|\beta'| = 0$, and (3.4) with $|\alpha| = |\beta| = 0$, we finally get

$$\lim_{\varepsilon\downarrow 0} t_{\varepsilon,u} = \int_{\mathbb{R}^n} e^{-ix'\cdot\xi}\lim_{\varepsilon\downarrow 0} r_{\varepsilon,u}\,dx', \qquad \lim_{\varepsilon\downarrow 0} r_{\varepsilon,u} = \int_{\mathbb{R}^n} e^{ix'\cdot\xi'}\lim_{\varepsilon\downarrow 0} q_{\varepsilon,u}\,\bar{d}\xi',$$

$$\lim_{\varepsilon\downarrow 0} q_{\varepsilon,u} = \int_{\mathbb{R}^n} e^{-ix''\cdot\xi'}\lim_{\varepsilon\downarrow 0} p_{\varepsilon,u}\,dx''$$

so that by (3.3),

$$Pu(x) \;=\; \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} e^{ix\cdot\xi}t_{\varepsilon,u}(x,\xi)\,\bar{d}\xi \;=\; \int_{\mathbb{R}^n} e^{ix\cdot\xi}\lim_{\varepsilon\downarrow 0} t_{\varepsilon,u}(x,\xi)\,\bar{d}\xi \;=\; (3.0)$$

which proves the claim. □

講義に対する感想

この講義のおかげで擬微分作用素に関する理論に入門することができるようになりました。特に振動積分と擬微分作用素のいろいろな理論が僕の専攻な偏微分方程式にも応用ができて、偏微分方程式を研究するもう一つの方法を習得するようになりました。講義の内容も満足でした。擬微分作用素の理論の準備としてフーリエ変換、超関数、振動積分の理論を順々に講義してくださって大体無理なく理解できました。講義の後半には擬微分作用素の偏微分方程式への応用、最後にはHörmanderの「解の特異性は方程式の主表象がつくるHamiltonianFlowに沿って、伝播する」という意味を持つ定理などの話があって少し難しかったですが、おもしろかったです。また毎回講義で講義と関連がある参考文献も紹介してくださって、かなり助けになりました。
最近入院されたという話を聞きましたが一日も早い快癒をお願っています。今後ともよろしくお願いいたします。

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