A Report on Differential Equations I

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Notations

We introduce some notations here. We denote the Lebesgue space by L^p and the corresponding norm as $\|\cdot\|_{L^p}$ for $1 \leqslant p \leqslant \infty$. Also we denote the Sobolev space by H^s with the norm $\|\cdot\|_{H^s}$ for nonnegative integer s which is defined by

$$\|f\|_{H^s} = \|\langle\xi\rangle^s \mathcal{F}f\|_{L^2}$$

where $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$ and $\mathcal{F}$ is the Fourier transform given by

$$\mathcal{F}f = \hat{f} = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx.$$

For the simplicity, we write $\|\cdot\| = \|\cdot\|_{L^2}$ and $\|\cdot\|_s = \|\cdot\|_{H^s}$.

Exercises

Exercise 1. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $f(x) \in C_0^\infty(\mathbb{R}^3)$. Show that

$$\text{(i)}\ \ \|f\|_{L^p} \leqslant C\|f\|^{\frac{6-p}{2p}}\|Df\|^{\frac{3(p-2)}{2p}}$$

$$\text{(ii)}\ \ \sup_x |f(x)| \leqslant C\|f\|^{\frac{1}{4}}\|D^2 f\|^{\frac{3}{4}}$$

for $2 \leqslant p \leqslant 6$.

Solution. (i) Since $f \in C_0^\infty$, we have

$$f(x_1,x_2,x_3) = \int_{-\infty}^{x_1} f_{x_1}(a,x_2,x_3)\,da = \int_{-\infty}^{x_2} f_{x_2}(x_1,b,x_3)\,db = \int_{-\infty}^{x_3} f_{x_3}(x_1,x_2,c)\,dc$$

so that

$$|f(x)| \leqslant \int_{\mathbb{R}} |f_{x_1}(a,x_2,x_3)|\,da, \quad |f(x)| \leqslant \int_{\mathbb{R}} |f_{x_2}(x_1,b,x_3)|\,db, \quad |f(x)| \leqslant \int_{\mathbb{R}} |f_{x_3}(x_1,x_2,c)|\,dc.$$

Therefore we get

$$|f(x)|^{\frac{3}{2}} \leqslant \left(\int_{\mathbb{R}} |f_{x_1}(a,x_2,x_3)|\,da\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f_{x_2}(x_1,b,x_3)|\,db\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f_{x_3}(x_1,x_2,c)|\,dc\right)^{\frac{1}{2}}. \tag{1}$$

Integrating both sides of (1) with respect to x_1, and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}} |f(x)|^{\frac{3}{2}}\, dx_1 \\
&\leqslant \left(\int_{\mathbb{R}} |f_{x_1}(a,x_2,x_3)|\, da\right)^{\frac{1}{2}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_{x_2}(x_1,b,x_3)|\, db\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f_{x_3}(x_1,x_2,c)|\, dc\right)^{\frac{1}{2}} dx_1 \\
&\leqslant \left(\int_{\mathbb{R}} |f_{x_1}(a,x_2,x_3)|\, da\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |f_{x_2}(x_1,b,x_3)|\, dbdx_1\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |f_{x_3}(x_1,x_2,c)|\, dcdx_1\right)^{\frac{1}{2}} \qquad (2)
\end{aligned}$$

Integrating both sides of (2) with respect to x_2, and applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)|^{\frac{3}{2}}\, dx_1dx_2 \\
&\leqslant \left(\int_{\mathbb{R}^2} |f_{x_2}(x_1,b,x_3)|\, dbdx_1\right)^{\frac{1}{2}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_{x_1}(a,x_2,x_3)|\, da\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |f_{x_3}(x_1,x_2,c)|\, dcdx_1\right)^{\frac{1}{2}} dx_2 \\
&\leqslant \left(\int_{\mathbb{R}^2} |f_{x_2}(x_1,b,x_3)|\, dbdx_1\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |f_{x_1}(a,x_2,x_3)|\, dadx_2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |f_{x_3}(x_1,x_2,c)|\, dcdx_1dx_2\right)^{\frac{1}{2}}. \qquad (3)
\end{aligned}$$

Integrating both sides of (3) with respect to x_3, and applying Cauchy-Schwarz inequality, we finally get

$$\begin{aligned}
&\|f\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} = \int_{\mathbb{R}^3} |f(x)|^{\frac{3}{2}}\, dx \\
&\leqslant \left(\int_{\mathbb{R}^3} |f_{x_1}(a,x_2,x_3)|\, dadx_2dx_3\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |f_{x_2}(x_1,b,x_3)|\, dbdx_1dx_3\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |f_{x_3}(x_1,x_2,c)|\, dcdx_1dx_2\right)^{\frac{1}{2}} \\
&\leqslant \|Df\|_{L^1}^{\frac{3}{2}}. \qquad (4)
\end{aligned}$$

Replacing $f \to f^4$ in (4), and using Cauchy-Schwarz inequality, we obtain

$$\|f\|_{L^6}^6 = \|f^4\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} \leqslant \|Df^4\|_{L^1}^{\frac{3}{2}} \leqslant \|4f^3Df\|_{L^1}^{\frac{3}{2}} \leqslant (4\|f^3\|_{L^2}\|Df\|_{L^2})^{\frac{3}{2}} = 8\|f\|_{L^6}^{\frac{9}{2}}\|Df\|_{L^2}^{\frac{3}{2}}$$

or

$$\int_{\mathbb{R}^3} |f(x)|^6\, dx \leqslant 8\left(\int_{\mathbb{R}^3} |f(x)|^6\, dx\right)^{\frac{3}{4}} \left(\int_{\mathbb{R}^3} |Df(x)|^2\, dx\right)^{\frac{3}{4}}. \qquad (5)$$

If $\|f\|_{L^6} = 0$ then (i) is trivial, so let $\|f\|_{L^6} \neq 0$. So from (5), we have

$$\left(\int_{\mathbb{R}^3} |f(x)|^6\, dx\right)^{\frac{1}{4}} \leqslant 8\left(\int_{\mathbb{R}^3} |Df(x)|^2\, dx\right)^{\frac{3}{4}}$$

so that

$$\int_{\mathbb{R}^3} |f(x)|^6\, dx \leqslant 8^4\left(\int_{\mathbb{R}^3} |Df(x)|^2\, dx\right)^3. \qquad (6)$$

Note that $p = \frac{6-p}{4}\cdot 2 + \frac{p-2}{4}\cdot 6$, $0 \leqslant \frac{6-p}{4}, \frac{p-2}{4} \leqslant 1$ and $\frac{6-p}{4} + \frac{p-2}{4} = 1$ for $2 \leqslant p \leqslant 6$. Thus by the Hölder inequality and (6), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |f(x)|^p\,dx &\leqslant \left(\int_{\mathbb{R}^3} |f(x)|^2\,dx\right)^{\frac{6-p}{4}} \left(\int_{\mathbb{R}^3} |f(x)|^6\,dx\right)^{\frac{p-2}{4}} \\ &\leqslant 8^{p-2}\left(\int_{\mathbb{R}^3} |f(x)|^2\,dx\right)^{\frac{6-p}{4}} \left(\int_{\mathbb{R}^3} |Df(x)|^2\,dx\right)^{\frac{3(p-2)}{4}} \\ &= 8^{p-2}\|f\|^{\frac{6-p}{2}}\|Df\|^{\frac{3(p-2)}{2}}\end{aligned}$$

so that

$$\|f\|_{L^p} \leqslant 8^{p-2}\|f\|^{\frac{6-p}{2p}}\|Df\|^{\frac{3(p-2)}{2p}}$$

for $\leqslant p \leqslant 6$ with $C = 8^{p-2}$ which proves (i).

(ii) Since $f = \mathcal{F}^{-1}\mathcal{F}f = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\widehat{f}(\xi)\,d\xi$, we have

$$|f(x)| \leqslant C\int_{\mathbb{R}^3} |\widehat{f}(\xi)|\,d\xi \leqslant C\left(\int_{|\xi|\leqslant R} |\widehat{f}(\xi)|\,d\xi + \int_{|\xi|\geqslant R} |\widehat{f}(\xi)|\,d\xi\right) \tag{7}$$

for any $R > 0$. For the first term of (7), the Cauchy-Schwarz inequality and the Plancheral theorem yields

$$\int_{|\xi|\leqslant R} |\widehat{f}(\xi)|\,d\xi \leqslant \left(\int_{|\xi|\leqslant R} d\xi\right)^{1/2}\left(\int_{|\xi|\leqslant R} |\widehat{f}(\xi)|^2\,d\xi\right)^{1/2} \leqslant CR^{3/2}\|\widehat{f}\| = CR^{3/2}\|f\|.$$

For the second term of (7), using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{|\xi|\geqslant R} |\widehat{f}(\xi)|\,d\xi &= \int_{|\xi|\geqslant R} \frac{1}{|\xi|^2}\cdot|\xi|^2|\widehat{f}(\xi)|\,d\xi \\ &\leqslant \left(\int_{|\xi|\geqslant R}\frac{d\xi}{|\xi|^4}\right)^{1/2}\left(\int_{|\xi|\geqslant R}|\xi|^4|\widehat{f}(\xi)|^2\,d\xi\right)^{1/2} \\ &\leqslant \left(C\int_R^\infty \frac{dr}{r^2}\right)^{1/2}\left(\int_{\mathbb{R}^3}\langle\xi\rangle^4|\widehat{f}(\xi)|^2\,d\xi\right)^{1/2} \\ &\leqslant CR^{-1/2}\|D^2 f\|.\end{aligned}$$

Summing up, we obtain

$$|f(x)| \leqslant CR^{3/2}\|f\| + CR^{-1/2}\|D^2 f\|. \tag{8}$$

If $\|f\| = 0$ then (ii) is trivial, so we assume $\|f\| \neq 0$ and let $R = \frac{\|D^2 f\|^{1/2}}{\|f\|^{1/2}}$. Then by (8), we finally get

$$\begin{aligned}|f(x)| &\leqslant C\frac{\|D^2 f\|^{3/4}}{\|f\|^{3/4}}\|f\| + C\frac{\|f\|^{1/4}}{\|D^2 f\|^{1/4}}\|D^2 f\| \\ &\leqslant C\|f\|^{1/4}\|D^2 f\|^{3/4}\end{aligned}$$

which proves (ii). □

Exercise 2. Consider the following initial value problem of Burgers equation:

$$\begin{cases} u_t + \left(\frac{1}{5}u^5\right)_x - \mu u_{xx} = 0, & t \geqslant 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{9}$$

where $\mu > 0$. Assume that if $u_0 \in H^3$ then there exists a solution of (9) $u \in C^0([0,\infty); H^3) \cap C^1([0,\infty); H^1)$. Prove that there exists a constant $C_0(u_0) > 0$ depending on only $\|u_0\|_1$ such that

$$\|u(t)\|_1^2 + \int_0^t \|u_x(\tau)\|_1^2 \, d\tau \leqslant C_0(u_0) \tag{10}$$

for all $t \geqslant 0$.

Solution. Multiplying u to the both sides of (9), we have

$$uu_t + u^5 u_x - \mu u u_{xx} = 0$$

so that

$$\frac{1}{2}(u^2)_t + \left(\int_0^u s^5 \, ds \right)_x - \mu u u_{xx} = 0. \tag{11}$$

Integrating both sides of (11) with respect to x, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2 \, dx + \int_{\mathbb{R}} \mu u_x^2 \, dx = 0. \tag{12}$$

Integrating both sides of (12) with respect to t, we obtain

$$\frac{1}{2}\|u(t)\|^2 + \mu \int_0^t \|u_x(\tau)\|^2 \, d\tau = \frac{1}{2}\|u_0\|^2. \tag{13}$$

Now we multiply $-u_{xx}$ to the both sides of (9). Then we get

$$-u_{xx}u_t - u^4 u_{xx} u_x + \mu u_{xx}^2 = 0. \tag{14}$$

Integrating both sides of (14) with respect to x, we have

$$\int_{\mathbb{R}} u_x u_{xt} \, dx - \int_{\mathbb{R}} u^4 u_{xx} u_x \, dx + \mu \int_{\mathbb{R}} u_{xx}^2 \, dx = 0$$

or

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + \mu \|u_{xx}\|^2 = \int_{\mathbb{R}} u^4 u_{xx} u_x \, dx. \tag{15}$$

Integrating both sides of (15) with respect to t, we obtain

$$\begin{aligned} \frac{1}{2}\|u_x(t)\|^2 + \mu \int_0^t \|u_{xx}(\tau)\|^2 \, d\tau &\leqslant \frac{1}{2}\|u_{0,x}\|^2 + \int_0^t \int_{\mathbb{R}} u^4 |u_{xx}||u_x| \, dx d\tau \\ &\leqslant \frac{1}{2}\|u_{0,x}\|^2 + \int_0^t \int_{\mathbb{R}} \frac{\mu}{2} u_{xx}^2 + \frac{1}{2\mu} u^8 u_x^2 \, dx d\tau. \end{aligned} \tag{16}$$

To estimate the last term of (16), we introduce the function Φ defined by

$$\Phi(u) = \int_0^u \int_0^r s^8 \, ds dr = \frac{u^{10}}{90}$$

so that $\Phi'(u) = u^9/9$ and $\Phi''(u) = u^8 \geqslant 0$. Multiplying $\Phi'(u)$ to the both sides of (9), we have

$$\Phi'(u)u_t + u^4 \Phi'(u) u_t - \mu \Phi'(u) u_{xx} = 0. \tag{17}$$

We note that $u^4\Phi'(u)u_t = \left(\int_0^u s^4\Phi'(s)\,ds\right)_x$. So integrating (17) with respect to x, we get

$$\frac{d}{dt}\int_{\mathbb{R}}\Phi(u)\,dx + \mu\int_{\mathbb{R}}\Phi''(u)u_x^2\,dx = 0. \tag{18}$$

Integrating (18) with respect to t we obtain

$$\int_{\mathbb{R}}\Phi(u(t))\,dx + \mu\int_0^t\int_{\mathbb{R}}\Phi''(u)u_x^2\,dxd\tau = \int_{\mathbb{R}}\Phi(u_0)\,dx. \tag{19}$$

Since $\int_{\mathbb{R}}\Phi(u(t))dx \geqslant 0$, we have

$$\int_0^t\int_{\mathbb{R}}u^8u_x^2\,dxd\tau \leqslant \frac{1}{90\mu}\int_{\mathbb{R}}u_0^{10}\,dx \leqslant \frac{C(u_0)}{\mu}. \tag{20}$$

Therefore from (16),

$$\frac{1}{2}\|u_x(t)\|^2 + \mu\int_0^t\|u_{xx}(\tau)\|^2\,d\tau \leqslant \frac{1}{2}\|u_{0,x}\|^2 + \frac{\mu}{2}\int_0^t\|u_{xx}\|^2\,d\tau + \frac{C(u_0)}{2\mu^2}$$

so that

$$\frac{1}{2}\left(\|u_x(t)\|^2 + \mu\int_0^t\|u_{xx}(\tau)\|^2\,d\tau\right) \leqslant \frac{1}{2}\|u_{0,x}\|^2 + \frac{C(u_0)}{2\mu^2} \leqslant C(u_0). \tag{21}$$

Thus by combining (13) and (21), we reach at

$$\|u(t)\|_1^2 + \int_0^t\|u_x(\tau)\|_1^2\,d\tau \ \leqslant\ C_0(u_0)$$

for all $t \geqslant 0$ which proves (10). □

Exercise 3. Consider the following initial value problem:

$$\begin{cases} v_t - u_x = 0 \\ u_t + \left(\frac{1}{v}\right)_x = \left(\frac{u_x}{v}\right)_x, & t \geqslant 0,\ x \in \mathbb{R}, \\ (v,u)(0,x) = (v_0,u_0)(x), & x \in \mathbb{R}. \end{cases} \tag{22}$$

Assume that there exists a solution (v,u) to (22) such that $v-1 \in C^0([0,\infty);H^3)\cap C^1([0,\infty);H^2)$, $\inf_{t,x} v > 0$ and $u \in C^0([0,\infty);H^3)\cap C^1([0,\infty);H^2)$ if $v_0 - 1, u_0 \in H^3$, $\inf_x v_0 > 0$. Prove that there exists constants $C_0(v_0,u_0), \underline{v}(v_0,u_0) > 0$ depending on only $\|(v_0-1,u_0)\|_1$ and $\inf v_0$ such that

$$\|(v-1,u)(t)\|_1^2 + \int_0^t(\|v_x(\tau)\|^2 + \|u_x(\tau)\|_1^2)\,d\tau \leqslant C_0(v_0,u_0) \tag{23}$$

and

$$\inf_{t\geqslant 0, x\in\mathbb{R}} v(t,x) \geqslant \underline{v}(v_0,u_0) \tag{24}$$

for all $t \geqslant 0$.

Solution. Multiplying u to the both sides of (22) and integrating with respect to x, we have

$$\int_{\mathbb{R}} uu_t + u\left(\frac{1}{v}-1\right)_x - \left(\frac{u_x}{v}\right)_x u\,dx = 0$$

so that

$$\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}u^2\,dx-\int_{\mathbb{R}}u_x\left(\frac{1}{v}-1\right)\,dx+\int_{\mathbb{R}}\frac{u_x^2}{v}\,dx=0$$

or

$$\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}u^2\,dx-\int_{\mathbb{R}}v_t\left(\frac{1}{v}-1\right)\,dx+\int_{\mathbb{R}}\frac{u_x^2}{v}\,dx=0 \tag{25}$$

where we used the relation $u_x = v_t$. Now we define Φ by

$$\Phi(v)=\int_1^v\left(1-\frac{1}{s}\right)\,ds=v-\log v-1$$

as Figure 1 below.

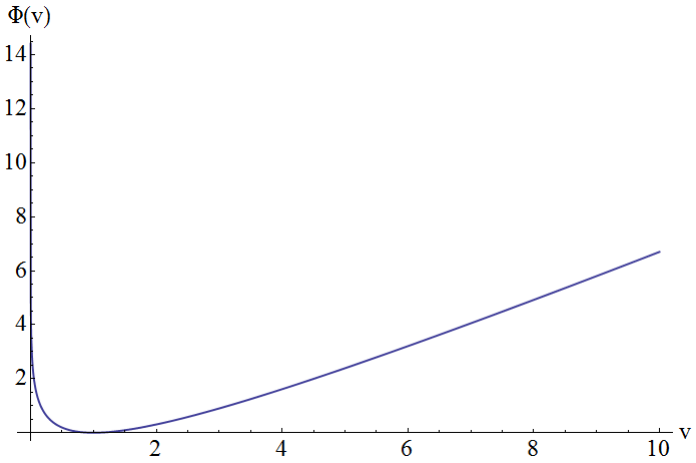

Figure 1: The graph of $\Phi(v)$

Since

$$-\int_{\mathbb{R}}v_t\left(\frac{1}{v}-1\right)\,dx=\int_{\mathbb{R}}v_t\Phi'(v)\,dx=\frac{d}{dt}\int_{\mathbb{R}}\Phi(v)\,dx,$$

(25) becomes

$$\frac{d}{dt}\left(\int_{\mathbb{R}}\frac{1}{2}u^2+\Phi(v)\,dx\right)+\int_{\mathbb{R}}\frac{u_x^2}{v}\,dx=0. \tag{26}$$

Also we note that $(v_t/v)_x = (\log v)_{tx} = (\log v)_{xt} = (v_x/v)_t$ and $(1/v)_x = -v_x/v^2$. Then the second equation of (22) becomes

$$\left(\frac{v_x}{v}\right)_t+\frac{v_x}{v^2}-u_t=0. \tag{27}$$

Multiplying (v_x/v) to the both sides of (27) and integrating with respect to x, we have

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}}\left(\frac{v_x}{v}\right)^2\,dx+\int_{\mathbb{R}}\frac{v_x^2}{v^3}\,dx-\int_{\mathbb{R}}\frac{v_x}{v}u_t\,dx=0. \tag{28}$$

Because we have

$$\begin{aligned}-\int_{\mathbb{R}}\frac{v_x}{v}u_t\,dx &= \int_{\mathbb{R}}-\left(\frac{v_x}{v}u\right)_t+\left(\frac{v_x}{v}\right)_t u\,dx=\int_{\mathbb{R}}-\left(\frac{v_x}{v}u\right)_t+\left(\frac{v_t}{v}\right)_x u\,dx\\ &= \int_{\mathbb{R}}-\left(\frac{v_x}{v}u\right)_t-\left(\frac{v_t}{v}\right)u_x\,dx=\int_{\mathbb{R}}-\left(\frac{v_x}{v}u\right)_t-\frac{u_x^2}{v}\,dx,\end{aligned}$$

from (28), we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}\left(\frac{v_x}{v}\right)^2 dx - \int_{\mathbb{R}} u\frac{v_x}{v}\,dx\right) + \int_{\mathbb{R}}\frac{v_x^2}{v^3}\,dx - \int_{\mathbb{R}}\frac{u_x^2}{v}\,dx = 0. \tag{29}$$

Multiplying $\frac{1}{2}$ to (29) and combining with (26), we get

$$\frac{d}{dt}\int_{\mathbb{R}}\left(\frac{1}{2}u^2 - \frac{1}{2}u\frac{v_x}{v} + \frac{1}{4}\left(\frac{v_x}{v}\right)^2 + \Phi(v)\right)dx + \int_{\mathbb{R}}\left(\frac{1}{2}\frac{u_x^2}{v} + \frac{1}{2}\frac{v_x^2}{v^3}\right)dx = 0. \tag{30}$$

Combining (26) and (30), we obtain

$$\int_{\mathbb{R}}\left(u^2 + \Phi(v) + \left(\frac{v_x}{v}\right)^2\right)dx + \int_0^t\int_{\mathbb{R}}\left(\frac{u_x^2}{v} + \frac{v_x^2}{v^3}\right)dxd\tau \leqslant C(v_0,u_0) \tag{31}$$

where $C(v_0,u_0)$ is a constant depending on $\inf v_0$, $\|v_0-1\|_1$ and $\|u_0\|$. Now we have to show that v is bounded above, and also bounded below. For the purpose, we introduce Kanel's method here. Define Ψ by

$$\Psi(v) = \int_1^v \frac{\Phi(s)^{1/2}}{s}\,ds = \int_1^v \frac{\sqrt{s-\log s-1}}{s}\,ds.$$

Then we can check that

$$\lim_{v\to+\infty}\Psi(v) = +\infty \quad \text{and} \quad \lim_{v\downarrow 0}\Psi(v) = -\infty \tag{32}$$

as shown in Figure 2.

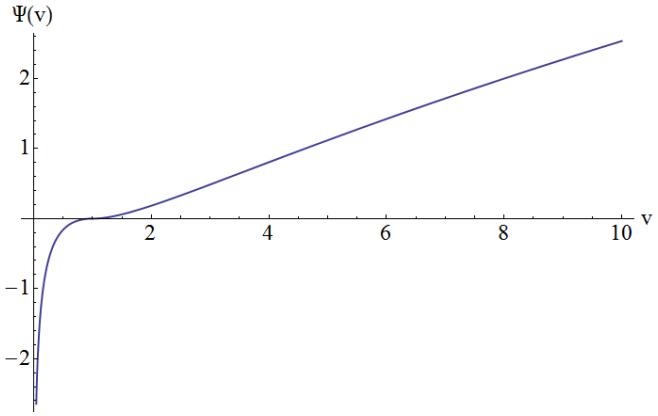

Figure 2: The graph of $\Psi(v)$

Since $\Psi(v(t,x)) = \int_{-\infty}^x \partial_y(\psi(v(t,y)))dy = \int_{-\infty}^x \Psi'(v)v_y dy$, using (31), we have

$$|\Psi(v(t,x))| \leqslant \int_{\mathbb{R}}\frac{\Phi(v)^{1/2}}{v}|v_x|\,dx \leqslant \frac{1}{2}\int_{\mathbb{R}}\left(\Phi(v) + \left(\frac{v_x}{v}\right)^2\right)dx \leqslant C(v_0,u_0). \tag{33}$$

Therefore by (32), there exist $\underline{v},\overline{v} > 0$ depending only on (v_0,u_0) such that

$$\underline{v} \leqslant v(t,x) \leqslant \overline{v} \tag{34}$$

for all $t \geqslant 0$ and $x \in \mathbb{R}$. This proves the inequality (24). Now we are going to complete the proof of (23). We note that (34) and (31) implies

$$\|(v-1)(t)\|_1^2 + \|u(t)\|^2 + \int_0^t \left(\|v_x(\tau)\|^2 + \|u_x(\tau)\|^2\right)d\tau \leqslant C(v_0,u_0) \tag{35}$$

for $t \geqslant 0$. From now on, we are going to prove the estimate for u_x. The second equation of (22) can be written as

$$u_t - \frac{1}{v^2}v_x - \frac{u_{xx}}{v} + \frac{u_x v_x}{v^2} = 0. \tag{36}$$

Multiplying $-u_{xx}$ to (36) and integrating with respect to x, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_x^2\,dx + \int_{\mathbb{R}} \frac{u_{xx}^2}{v}\,dx = \int_{\mathbb{R}} -u_{xx}\frac{v_x}{v^2}\,dx + \int_{\mathbb{R}} \frac{u_{xx}u_x v_x}{v^2}\,dx. \tag{37}$$

Now we estimate the right hand side of (37). For the first term, using (34), we have

$$\begin{aligned}\left|\int_{\mathbb{R}} u_{xx}\frac{v_x}{v^2}\,dx\right| &\leqslant C\int_{\mathbb{R}} |u_{xx}||v_x|\,dx \\ &\leqslant \int_{\mathbb{R}} \varepsilon u_{xx}^2 + \frac{C}{\varepsilon}v_x^2\,dx \end{aligned}\tag{38}$$

for $\varepsilon > 0$. For the second term, using (34), we have

$$\begin{aligned}\left|\int_{\mathbb{R}} \frac{u_{xx}u_x v_x}{v^2}\,dx\right| &\leqslant C\int_{\mathbb{R}} |u_{xx}||u_x||v_x|\,dx \\ &\leqslant \int_{\mathbb{R}} \varepsilon u_{xx}^2\,dx + \int_{\mathbb{R}} \frac{C}{\varepsilon}u_x^2 v_x^2\,dx \\ &\leqslant \varepsilon\|u_{xx}\|^2 + \frac{C}{\varepsilon}\sup_{x\in\mathbb{R}}|u_x|^2\|v_x\|^2 \\ &\leqslant \varepsilon\|u_{xx}\|^2 + \frac{C}{\varepsilon}\|u_x\|\|u_{xx}\| \\ &\leqslant 2\varepsilon\|u_{xx}\|^2 + \frac{C}{\varepsilon^3}\|u_x\|^2 \end{aligned}\tag{39}$$

for $\varepsilon > 0$ where we used the Sobolev inequality. Therefore combining (38), (39) and (37), we finally obtain

$$\|u_x(t)\|^2 + \int_0^t \|u_{xx}(\tau)\|^2\,d\tau \leqslant C(v_0, u_0). \tag{40}$$

Thus by (35) and (40), we have

$$\|(v-1, u)(t)\|_1^2 + \int_0^t (\|v_x(\tau)\|^2 + \|u_x(\tau)\|_1^2)\,d\tau \leqslant C_0(v_0, u_0)$$

for all $t \geqslant 0$. This completes the proof of Exercise 3. □

References

[1] A. Matsumura and K. Nishihara, *A Global Solution to Nonlinear Partial Differential Equations*, Nihon Hyouronsha, 2004.